Homework 8

Revised version of solutions by Aja Johnson

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1

Theorem. *Let f be defined for all real x and suppose that $|f(x)-f(y)| \leq (x-y)^2$ for all real x and y. Then f is constant.*

Proof. Algebraically manipulating the given inequality yields for all $x \neq y$

$$\left|\frac{f(x)-f(y)}{x-y}\right| \leq \quad |x-y|, \text{ or equivalently}$$

$$-|x-y| \leq \frac{f(x)-f(y)}{x-y} \leq |x-y|.$$

Since $\lim_{y \to x} |x-y| = 0$, the Squeeze Theorem implies $f'(x) = 0$ for all $x \in \mathbb{R}$. By Theorem 5.11(b), f is constant over $\mathbb{R}$. □

3

Theorem. *Suppose g is a real function on $\mathbb{R}$, with bounded derivative (say $|g'| \leq M$). Fix $\epsilon > 0$ and define $f(x) = x + \epsilon g(x)$. Then f is one-to-one if ϵ is small enough.*

Proof. Take $0 < \epsilon < \frac{1}{M}$. Since $g'(x) \geq -M$ for all x, we conclude that $f'(x) = 1 + \epsilon g'(x) > 0$ for all $x \in \mathbb{R}$ whence f is strictly increasing and hence one-to-one. □

4

Theorem. *If $C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0$, where $C_0, \ldots, C_n$ are real constants, then the equation $C_0 + C_1 x + \cdots + C_{n-1}x^{n-1} + C_n x^n = 0$ has at least one real root between 0 and 1.*

Proof. Let $f(x) = C_0 x + \frac{C_1}{2}x^2 + \cdots + \frac{C_{n-1}}{n}x^n + \frac{C_n}{n+1}x^{n+1}$. Since f is a polynomial, it is continuous and differentiable. Since $f(0) = f(1) = 0$ by hypothesis, the mean value theorem guarantees that the equation $f'(x) = 0$ has a solution in the interval $(0, 1)$, as desired. □

5

Theorem. *Suppose that f is defined and differentiable for every $x > 0$, and $f'(x) \to 0$ as $x \to \infty$. Let $g(x) = f(x+1) - f(x)$. Then $g(x) \to 0$ as $x \to \infty$.*

Proof. Let $\epsilon > 0$. Apply the hypothesis to find a number N such that $|f'(t)| < \epsilon$ whenever $t \geq N$.

Now suppose $x \geq N$ and apply the Mean Value Theorem to find a number $t \in (x, x+1)$ such that $g(x) = f(x+1) - f(x) = g'(t)$. Since $t > N$, we have $|g(x)| = |f'(t)| < \epsilon$. □

7

Theorem. *Suppose that $f'(x)$, $g'(x)$ exist, $g'(x) \neq 0$, and $f(x) = g(x) = 0$. Then $\lim_{t \to x} \dfrac{f(t)}{g(t)} = \dfrac{f'(x)}{g'(x)}$.*

Proof. Since $g'(x) \neq 0$, we have $g(t) \neq g(x)$ for t sufficiently close to but unequal to x. For such t, we have

$$\frac{f(t)}{g(t)} = \frac{f(t) - f(x)}{g(t) - g(x)} = \frac{\frac{f(t)-f(x)}{t-x}}{\frac{g(t)-g(x)}{t-x}}.$$

It only remains to take the limit as $t \to x$. □

8

Theorem. *Suppose that f' is continuous on $[a, b]$ and $\epsilon > 0$. Then there exists a $\delta > 0$ such that for $x, t \in [a, b]$, if $0 < |t - x| < \delta$, then $\left| \dfrac{f(t) - f(x)}{t - x} - f'(x) \right| < \epsilon$.*

Proof. Let $\epsilon > 0$. Since $[a, b]$ is compact, by Theorem 4.19 f' is uniformly continuous. Pick $\delta > 0$ such that $|f'(y) - f'(x)| < \epsilon$ whenever $0 < |y - x| < \delta$.

Now suppose $x, t \in [a, b]$ with $0 < |t - x| < \delta$. Apply the Mean Value Theorem to find $c \in (x, t)$ such that $f(t) - f(x) = (t - x)f'(c)$. Since $0 < |c - x| < \delta$, we have $|\frac{f(t)-f(x)}{t-x} - f'(x)| = |f'(c) - f'(x)| < \epsilon$, as required. □

Question. *Does this hold for vector-valued functions too?*

Answer. Yes. Consider the vector valued function $F = (f_1, \ldots, f_n) : [a, b] \to \mathbb{R}^n$. Since F' is continuous, each component function f_j' is uniformly continuous. Let $\epsilon > 0$. Pick $\delta > 0$ such that for each $j = 1 \ldots n$ and $p, q \in [a, b]$, if $|p - q| < \delta$ then $|f_j'(p) - f_j'(q)| < \frac{\epsilon}{n}$.

Now suppose $x, t \in [a, b]$, with $0 < |t - x| < \delta$. For each $1 \leq j \leq n \in \mathbb{N}$, apply the Mean Value Theorem to get $c_j \in (x, t)$ such that $f_j(t) - f_j(x) = (t - x)f_j'(c_j)$

Certainly $0 < |c_j - x| < |t - x| < \delta$ for each such j, so

$$\left|\frac{F(t) - F(x)}{t - x} - F'(x)\right| = \sqrt{\sum_{j=1}^{n} \left(f_j'(c_j) - f_j(x)\right)^2} \leq \sqrt{n\frac{\epsilon^2}{n^2}} = \frac{\epsilon}{\sqrt{n}} < \epsilon$$

9

Question. *Let f be a continuous function on $\mathbb{R}$, of which it is know that $f'(x)$ exists for all $x \neq 0$ and that $f'(x) \to 3$ as $x \to 0$. Does it follow that $f'(0)$ exists?*

Answer. By L'Hopital's Rule, we have $\lim_{t\to 0} \frac{f(t) - f(0)}{t - 0} = \lim_{t\to 0} \frac{f'(t)}{1}$. But it is given that $\lim_{t\to 0} f'(t) = 3$. Thus $f'(0) = 3$.

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Theorem. *Suppose that f is defined in a neighborhood of x, and suppose that $f''(x)$ exists. Then $\lim_{h\to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x)$.*

Proof. By definition $f'(x) = \lim_{t\to x} \frac{f(t) - f(x)}{t - x}$. Substituting $h = t - x$ into this expression yields $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$. Substituting $h = x - t$ into this expression yields $f'(x) = \lim_{h\to 0} \frac{f'(x-h) - f'(x)}{-h}$. So
$f''(x) = \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h\to 0} \frac{f'(x-h) - f'(x)}{-h}$. Thus using L'Hopital's Rule with respect to h, and algebraic manipulation yields

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} &= \lim_{h\to 0} \frac{f'(x+h) - f'(x-h)}{2h} \\
&= \frac{1}{2}\left(\lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h} + \lim_{h\to 0} \frac{f'(x-h) - f'(x)}{-h}\right) \\
&= \frac{1}{2}(f''(x) + f''(x)) = f''(x).
\end{aligned}$$

□

Theorem. *The above limit may exist even if $f''(x)$ does not.*

Proof. Define $f : \mathbb{R} \to \mathbb{R}$ by taking $f(x)$ to be 1 when $x > 0$, to be 0 when $x = 0$, and to be -1 when $x < 0$. Then $f''(0)$ cannot exist since f is discontinuous at 0. However at $x = 0$, the limit from Part (a) reduces to $\lim_{h\to 0} \frac{f(h) + f(-h) - 2f(0)}{h^2} = 0$. □

12

Question. *If $f(x) = |x|^3$, compute $f'(x)$ and $f''(x)$ for all real x.*

Answer. Note that

$$f(x) = |x|^3 = \begin{cases} x^3 & x \geq 0 \\ -x^3 & x < 0. \end{cases}$$

Thus if $x > 0$, then $f'(x) = 3x^2$ while $f''(x) = 6x$.

Similarly, for $x < 0$, we have $f'(x) = -3x^2$ while $f''(x) = -6x$.

Because of the change of formula for f at 0, derivatives at that point must be evaluated from the definition.

$$f'(0) = \lim_{h\to 0} \frac{f(h)}{h} = \lim_{h\to 0} \frac{|h|^3}{h} = \lim_{h\to 0} h|h| = 0, \text{ and}$$

$$f''(0) = \lim_{h\to 0} \frac{f'(h)}{h} = \lim_{h\to 0} \frac{3h|h|}{h} = \lim_{h\to 0} 3|h| = 0, \text{ but}$$

$$f'''(0) = \lim_{h\to 0} \frac{f''(h)}{h} = \lim_{h\to 0} \frac{6|h|}{h}, \text{ which does not exist.}$$

25

Suppose that f is twice differentiable on $[a, b]$, $f(a) < 0$, $f(b) > 0$, $f'(x) \geq \delta > 0$, and $0 \leq f''(x) \leq M$ for all $x \in [a, b]$. Let ξ be the unique point in (a, b) at which $f(\xi) = 0$.

a

Question. *Choose $x_i \in (\xi, b)$, and define $\{x_n\}$ by $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. What does this mean geometrically in terms of a tangent to the graph of f?*

Answer. First note that $f'(x_n)$ is the slope of the tangent line to the curve $f(x)$ at the point $(x_n, f(x_n))$. Using the point slope formula to find where this tangent line intersects the x-axis yields $(x_{n+1}, 0)$ where $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$.

b

Theorem. *For each $n \in J$, we have*
(i) $x_n \geq \xi$, (ii) $f(x_n) \geq 0$, and (iii) $x_{n+1} \leq x_n$.

Proof. Note that (ii) always follows from (i) because f is increasing, and (iii) always follows from (ii) by the formula for x_{n+1} and the fact that $f' > 0$. Since (i) is true for $n = 1$ by assumption, we can complete the proof by showing that $x_n \geq \xi$ implies $x_{n+1} \geq \xi$.

So assume that $x_n \geq \xi$ for some fixed n, and set $g(x) = f(x) - (f(x_n) + f'(x_n)(x - x_n))$. This is the curve minus the tangent line through the point

$(x_n, f(x_n))$. By Theorem 5.3, $g'(x) = f'(x) - f'(x_n)$ and $g''(x) = f''(x) \geq 0$. Combining this with the fact that $g'(x_n) = 0$ gives that $g'(x) \leq 0$ for x to the left of x_n and $g'(x) \geq 0$ for x to the right of x_n. Combining this with the fact that $g(x_n) = 0$ gives that $g(x) \geq 0$ over its whole domain. Thus the tangent line through $(x_n, f(x_n))$ lies on or below the curve $f(x)$ and $x_{n+1} \geq \xi$ as desired. □

Theorem. *Then* $\lim_{n\to\infty} x_n = \xi$.

Proof. Since $\{x_n\}$ is non-increasing and bounded, it is convergent. Let $\lim_{n\to\infty} x_n = c$. Then $\lim_{n\to\infty} x_{n+1} = c$ also. So

$$\begin{aligned} \lim_{n\to\infty} x_{n+1} &= \lim_{n\to\infty} \left(x_n - \frac{f(x_n)}{f'(x_n)} \right) \\ c &= c - \frac{f(c)}{f'(c)}. \end{aligned}$$

This implies that $\frac{f(c)}{f'(c)} = 0$, which means that $f(c) = 0$. But ξ is the unique point in (a, b) such that $f(\xi) = 0$. Thus $c = \xi$ and $\lim_{n\to\infty} x_n = \xi$. □

c

Theorem. *Then* $x_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2$ *for some* $t_n \in (\xi, x_n)$.

Proof. (Thanks to Stephen Bismarck.) Applying Taylor's Theorem to f with $n = 2$, $\alpha = x_n$ and $\beta = \xi$ yields

$$f(\xi) = f(x_n) + f'(x_n)(\xi - x_n) + \frac{f''(t_n)}{2}(\xi - x_n)^2$$

for some t_n between ξ and x_n. Recalling that $f(\xi) = 0$ and rearranging this equation yields

$$x_n - \frac{f(x_n)}{f'(x_n)} - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2,$$

which is what we wanted to show. □

d

Theorem. *If* $A = \frac{M}{2\delta}$, *then* $0 \leq x_{n+1} - \xi \leq \frac{1}{A}[A(x_1 - \xi)]^{2^n}$.

Proof. By part (b) of this exercise $\xi \leq x_{n+1}$, so the first inequality holds for all n. The second inequality is obviously true when $n = 0$. Proceeding by induction, assume that $x_n - \xi \leq \frac{1}{A}[A(x_1 - \xi)]^{2^{n-1}}$. Then by part (c) of this exercise, the inductive hypothesis, the fact that $f'(x) > 0$ which implies that

$\frac{1}{f'(x)} < \frac{1}{\delta}$, and $0 \leq f''(x) \leq M$ for all $x \in [a,b]$,

$$\begin{aligned} x_{n+1} - \xi &= \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2 \\ &\leq A\left(\frac{1}{A}[A(x_1 - \xi)]^{2^{n-1}}\right)^2 \\ &\leq A\frac{1}{A^2}[A(x_1 - \xi)]^{2^{n-1}\cdot 2} \\ &\leq \frac{1}{A}[A(x_1 - \xi)]^{2^n}. \end{aligned}$$

□

e

Theorem. *Newton's Method is equivalent to finding a fixed point of the function g defined by $g(x) = x - \frac{f(x)}{f'(x)}$.*

Proof. Newton's method is equivalent to finding an x such that $f(x) = 0$, i.e. $\frac{f(x)}{f'(x)} = 0$. This is equivalent to saying that $g(x) = x - \frac{f(x)}{f'(x)} = x$, i.e. finding a fixed point of $g(x)$. □

Question. *How does $g'(x)$ behave near ξ?*

Answer. By Theorem 5.3, $g'(x) = \frac{f(x)f''(x)}{f'(x)f'(x)}$, so $\lim_{x\to\xi} g'(x) = g'(\xi) = 0$.

f

Question. *What happens when $f(x) = x^{\frac{1}{3}}$ on $(-\infty, \infty)$ and Newton's Method is applied?*

Answer. We have $\dfrac{f(x)}{f'(x)} = \dfrac{3x^{\frac{1}{3}}}{x^{\frac{-2}{3}}} = 3x$, so $x_{n+1} = x_n - 3x_n = -2x_n$ for all n. Thus, no matter what starting guess x_1 is chosen, the iterates (x_n) will diverge.