Homework 10 - Chapter 7 of Rudin

Revised version of solutions by Aja Johnson

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1

Theorem. *Every uniformly convergent sequence of bounded functions is uniformly bounded.*

Proof. Let (f_n) be a uniformly convergent sequence of bounded functions on a space X. Pick an integer N such that if $m, n \geq N$, then $|f_n(x) - f_m(x)| < 1$ for all $x \in X$. This N is guaranteed to exist by the Cauchy criterion for uniform convergence. By hypothesis, each f_n is bounded. By definition this means that for each $n \in \mathbb{N}$ there exists an $M_n \in \mathbb{R}$ such that $|f_n(x)| \leq M_n$ for all $x \in X$. Thus, if $n \geq N$, then

$$|f_n(x)| = |f_n(x) - f_N(x) + f_N(x)| \leq |f_n(x) - f_N(x)| + |f_N(x)| < 1 + M_N.$$

Let $M = \max\{1 + M_N, M_1, M_2, \ldots, M_{N-1}\}$. Then$|f_n(x)| \leq M$ for all $n \in \mathbb{N}$ and all $x \in X$. □

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Theorem. *If (f_n) and (g_n) converge uniformly on a set E, then $(f_n + g_n)$ converges uniformly on E.*

Proof. Let $\epsilon > 0$ and write f, g for the uniform limits of $(f_n), (g_n)$ respectively. Apply the definition of uniform convergence to choose $N_1, N_2 \in \mathbb{N}$, such that if $n \geq N_1, N_2$, then $|f_n(x) - f(x)| < \frac{\epsilon}{2}$ and
$|g_n(x) - g(x)| < \frac{\epsilon}{2}$, respectively, for all $x \in E$. Take $N = \max\{N_1, N_2\}$.

Suppose $n \geq N$ and $x \in E$. Then we have

$$|f_n(x) + g_n(x) - (f(x) + g(x))| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

as desired. □

Recall that if $h : E \to \mathbb{R}$ is a bounded function, then we write $||h|| :=$ $\sup\{|h(x)| : x \in E\}$. *(If more than one domain is being considered, we write $||h||_E$ to avoid ambiguity, but this is not usually necessary.) This notation will be used below. It saves us from writing "for all x in E" over and over again.*

Theorem. *If (f_n) and (g_n) are sequences of bounded functions that converge uniformly on a set E, then $(f_n g_n)$ converges uniformly on E.*

Proof. Let $\epsilon > 0$ and write f, g for the uniform limits of the given sequences of functions. Apply Exercise 1 to get a single constant M satisfying $||f_n|| \leq M$ and $||g_n|| \leq M$ for all $n \in J$. It follows from (even pointwise) convergence that $||f|| \leq M$ and $||g|| \leq M$ as well. Now apply the uniform convergence hypothesis to find $N \in J$ so that $||f_n - f|| < \frac{\epsilon}{2M+1}$ and $||g_n - g|| < \frac{\epsilon}{2M+1}$ for all $n \geq N$.

Let $n \geq N$. Then

$$||f_n g_n - fg|| \leq ||f_n|| ||g_n - g|| + ||g|| ||f_n - f|| \leq \frac{2M\epsilon}{2M+1} < \epsilon.$$

By definition, this means $(f_n g_n)$ converges uniformly to fg as desired. □

3

Question. *Construct sequences (f_n) and (g_n) which converge uniformly on some set E, but $(f_n g_n)$ does not converge uniformly on E.*

Answer. Let $f_n(x) = g_n(x) = x + \frac{1}{n}$ and $f(x) = g(x) = x$. Fix $\epsilon > 0$. Choose $N > \frac{1}{\epsilon}$. If $n \geq N$, then for all $x \in E$, we have

$$|f_n(x) - f(x)| = \frac{1}{n} < \epsilon.$$

Thus (f_n) converges uniformly to f. Likewise (g_n) converges uniformly to g.

It remains to show that $(f_n g_n)$ does not converge uniformly. If it were to converge to some function uniformly that function would be $f(x)g(x) = x^2$.Note that $f_n(x)g_n(x) = (x + \frac{1}{n})^2 = x^2 + \frac{2x}{n} + \frac{1}{n^2}$. Pick $\epsilon = 1$. Let N be given. Take $n = x = N$. Then

$$|f_n(x)g_n(x) - f(x)g(x)| = |x^2 + \frac{2x}{n} + \frac{1}{n^2} - x^2| = \left|\frac{2N^2 + 1}{N^2}\right| > 1.$$

Thus $(f_n g_n)$ does not converge uniformly.

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Theorem. *Let*

$$f_n(x) = \begin{cases} 0 & x < \frac{1}{n+1} \\ \sin^2 \frac{\pi}{x} & \frac{1}{n+1} \leq x \leq \frac{1}{n} \\ 0 & \frac{1}{n} < x. \end{cases}$$

Then (f_n) converges to a continuous function, but not uniformly.

Proof. Let $\epsilon > 0$. Fix $x \in \mathbb{R}$. Let $N > \frac{1}{x} \in \mathbb{N}$. (When $x \leq 0$, we take $N = 1$.) If $n \geq N$, then

$$|f_n(x) - 0| = 0 < \epsilon.$$

Thus for each $x \in \mathbb{R}$, the numerical sequence $(f_n(x))$ converges to 0.

Now we show that (f_n) does not converge uniformly. If it were to converge to some function uniformly that function would be 0. Pick $\epsilon = \frac{1}{2}$. Let N be given. Take $n = N$ and $x = \frac{1}{N+\frac{1}{2}}$. Then

$$|f_n(x) - 0| = \left|\sin^2(N + \frac{1}{2})\pi\right| = 1 > \frac{1}{2}.$$

Thus (f_n) does not converge uniformly. □

Theorem. *Pointwise absolute convergence for a series of functions does not imply uniform convergence.*

Proof. By definition, (pointwise or uniform) convergence of the series $\sum f_n$ is equivalent to that type of convergence for its sequence (s_n) of partial sums. Fix $x \in \mathbb{R}$. As shown above, there is an integer N (depending on x) so that $f_n(x) = 0$ for all $n \geq N$. But this means $s_n(x) = s_N(x)$ for all $n \geq N$, whence $\lim_{n\to\infty} s_n(x) = s_N(x)$, and the series $\sum f_n(x)$ converges to $s_N(x)$ by definition. Since $f_n(x) \geq 0$ for all $n \in \mathbb{N}$ and all $x \in E$, this convergence is absolute by default.

Suppose $\sum f_n$ converged uniformly to some function s. Then the sequence $(f_n) = (s_n - s_{n-1})$ would converge uniformly to $s - s = 0$. Since the first part of this problem showed that (f_n) does not converge uniformly, we conclude that the series $\sum f_n$ cannot converge uniformly either. □

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Theorem. *For $n \in J$ and $x \in \mathbb{R}$, put $f_n(x) = \frac{x}{1+nx^2}$. Then (f_n) converges uniformly to a function f, and $f'(x) = \lim_{n\to\infty} f_n'(x)$ is correct if $x \neq 0$, but false if $x = 0$.*

Proof. We first show that the sequence (f_n) converges uniformly to $f \equiv 0$, i.e., that $\lim_{n\to\infty} \|f_n\| = 0$. One can get a reasonable upper bound for $\|f_n\|$ by separately bounding $f_n(x)$ for x close to, and far away from, 0, but it is easier to use calculus. A direct computation shows that the only positive critical point of f_n is at $\frac{1}{n}$ and that in fact $f_n' > 0$ on the interval $(0, \frac{1}{n})$ while $f_n' < 0$ on $(\frac{1}{n}, \infty)$. In view of the mean value theorem, this means that $0 \leq f(x) \leq f(\frac{1}{n}) = \frac{2}{n}$ for all $x \geq 0$. Since all these functions are odd ($f_n(-x) = -f_n(x)$ for all n and all x), we conclude that $\lim \|f_n\| = \lim \frac{2}{n} = 0$, as desired.

The statements concerning $\lim_{n\to\infty} f_n'(x)$ are also direct computations. □

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Theorem. *Let (f_n) be a sequence of continuous functions which converges uniformly to a function f on a set E. Then $\lim_{n\to\infty} f_n(x_n) = f(a)$ for every sequence of points $x_n \in E$ such that $x_n \to a$, and $a \in E$.*

Proof. Let $\epsilon > 0$. Apply the uniform convergence hypothesis to choose $N_1 \in \mathbb{N}$ such that $\|f_n - f\| < \frac{\epsilon}{2}$ for all $n \geq N$. Theorem 7.12 tells us f is continuous. Then by definition a $\delta > 0$ can be chosen such that $|f(y) - f(a)| < \epsilon$ whenever $|y-a| < \delta$. Apply the definition of convergence to the numerical sequence (x_n) to choose $N_2 \in \mathbb{N}$ such that $|x_n - a| < \delta$ whenever $n \geq N_2$. Take $N = \max\{N_1, N_2\}$.

Suppose $n \geq N$. Then

$$|f_n(x_n) - f(a)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(a)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus $\lim_{n\to\infty} f_n(x_n) = f(a)$. □

Question. *Is the converse true?*

Answer. No. Let $E = \mathbb{Z}$ and $f_n(x) = 1$ if $x = n$ and $f_n(x) = 0$ otherwise. All functions from $\mathbb{Z}$ to $\mathbb{R}$ are continuous, so each f_n is continuous. Also, it is clear that (f_n) converges to 0 pointwise.

It remains to show that (f_n) does not converge uniformly. If it were to converge to some function uniformly that function would be $f(x) \equiv 0$. Pick $\epsilon = \frac{1}{2}$. Let N be given. Take $n = x = N$. Then $|f_n(x) - f(x)| = f_N(N) = 1 > \frac{1}{2}$, so (f_n) does not converge uniformly.

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Theorem. *Suppose that (f_n) and (g_n) are defined on E, $\sum f_n$ has uniformly bounded partial sums, $g_n \to 0$ uniformly on E, and $g_1(x) \geq g_2(x) \geq g_3(x) \geq \cdots$ for every $x \in E$. Then $\sum f_n g_n$ converges uniformly on E.*

Proof. This is a matter of adapting the proof of Dirictlet's Theorem 3.42 by replacing "a, b" by "f, g" and sprinkling in the term "uniformly". □

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Question. *Suppose f is a real continuous function on $\mathbb{R}$, $f_n(t) = f(nt)$ for $n = 1, 2, 3, \cdots$, and (f_n) is equicontinuous on $[0,1]$. What conclusion can you draw about f?*

Answer. f must be constant on the interval $[0, \infty)$.

Proof. The condition is sufficient because the equicontinuity hypothesis has no bearing on the values of $f(x)$ for $x < 0$. (This was pointed out by Ms. Ulrich.)

To establish necessity, fix $t > 0$. Let $\epsilon > 0$, and take $\delta > 0$ to be the corresponding number guaranteed by equicontinuity. Choose $n > \frac{t}{\delta}$. Then $\frac{t}{n} < \delta$ so $|f(t) - f(0)| = |f_n(\frac{t}{n}) - f(0)| < \epsilon$. The arbitrariness of ϵ means $f(t) = f(0)$ as desired. □

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Theorem. *Suppose that (f_n) is an equicontinuous sequence of functions on a compact set K, and (f_n) converges pointwise on K. Then (f_n) converges uniformly on K.*

The following argument is similar to the last few paragraphs in the proof of the Arzela-Ascoli Theorem 7.25. One can also use the result *of that theorem to establish the assertion of this problem.*

Proof. (outline) Write f for the pointwise limit of (f_n). Use the hypotheses in conjunction with the inequality

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)|$$

to establish uniform continuity of f.

Now, let $\epsilon > 0$ and take $\delta > 0$ to be the corresponding number guaranteed by equicontinuity.

Temporarily fix $a \in K$, and apply the inequality

$$|f_n(x) - f(x)| \leq |f_n(x) - f_n(a)| + |f_n(a) - f(a)| + |f(a) - f(x)|$$

to find a positive integer M_a such that $|f_n(x) - f(x)| < \epsilon$ whenever $n \geq N$ and $x \in N_\delta(a)$.

Finally, apply compactness to find finitely many points $a_1, \cdots, a_m$ so that the neighborhoods $N_\delta(a_1), \cdots, N_\delta(a_m)$ cover K. Take $M = \max\{M_{a_i}\}$ and check that $|f_n(x) - f(x)| < \epsilon$ for all $n \geq M$ and all $x \in K$. □

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Theorem. *Let (f_n) be a uniformly bounded sequence of functions which are Riemann-integrable on $[a, b]$, and put $F_n(x) = \int_a^x f_n(t)\, dt$ for $a \leq x \leq b$. Then there exists a subsequence (F_{n_k}) which converges uniformly on $[a, b]$.*

Proof. Choose a number M so that $\|f_n| \leq M$ for each $n \in J$. For each $x \in [a, b]$ and $n \in J$, we have $|F_n(x)| \leq \int_a^x |f_n| \leq (x - a)M \leq (b - a)M$, so the (F_n) are uniformly (and hence pointwise) bounded.

Moreover, for each $x, y \in [a, b]$, and $n \in J$, we have

$$|F_n(y) - F_n(x)| = |\int_x^y f_n| \leq M|y - x|,$$

establishing equicontinuity of the family $\{f_n\}$. The proof is completed by appealing to the Arzela-Ascoli Theorem 7.25. □