Homework 7

Revised version of solutions by Aja Johnson

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Problems are presented in rough order of increasing difficulty.

As mentioned in class, *to prove* a statement of type "for all $\epsilon > 0 \ldots$", we start with a "fixed but arbitrary" $\epsilon > 0$. When we are *given* a statement of type "for all $\epsilon > 0 \ldots$", we wait until we have a "particular" ϵ in mind and then invoke .... The bottom line is that the phrase "for all $\epsilon > 0$" should seldom appear in a proof.

11a

Theorem. *Suppose f is a uniformly continuous mapping of a metric space X into a metric space Y. If $\{x_n\}$ is a Cauchy sequence in X, then $\{f(x_n)\}$ is a Cauchy sequence in Y.*

Proof. Fix $\epsilon > 0$. Apply the uniform continuity hypothesis to choose a $\delta > 0$ such that $d(f(x), f(y)) < \epsilon$ whenever $d_X(x, y) < \delta$. Since $\{x_n\}$ is Cauchy there exists an $N \in \mathbb{J}$ such that $d(x_n, x_m) < \delta$ whenever $n, m \geq N$.

Suppose $n, m \geq N$. Then $d(x_n, x_m) < \delta$ whence $d(f(x_n), f(x_m)) < \epsilon$ as desired. □

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Theorem. *Suppose X, Y, Z are metric spaces, f maps X into Y, g maps the range of f, $f(X)$, into Z, and h is the mapping of X into Z defined by $h(x) = g(f(x))$. If f and g are both uniformly continuous, then h is also uniformly continuous.*

Proof. Fix $\epsilon > 0$. Since g is uniformly continuous, there exists an $\eta > 0$ such that $d(g(y_1), g(y_2)) < \epsilon$ whenever $y_1, y_2 \in \text{Ran} f$ satisfy $d(y_1, y_2) < \eta$. Since f is uniformly continuous, there exists a $\delta > 0$ such that $d(f(x_1), f(x_2)) < \eta$ whenever $d_X(x_1, x_2) < \delta$.

Suppose $d_X(x_1, x_2) < \delta$. Then $d(f(x_1), f(x_2)) < \eta$ whence $d(h(x_1), h(x_2)) = d(g(f(x_1)), g(f(x_2))) < \epsilon$. Thus h is uniformly continuous as desired. □

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Theorem. *Let $I = [0, 1]$ be the closed unit interval. Suppose that f is a continuous mapping of I into I. Then $f(x) = x$ for at least one $x \in I$.*

Proof. Apply the intermediate value theorem to the the function $g : I \to I$ defined by $g(x) = f(x) - x$. $\square$

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Question. *What discontinuities do the functions $[x]$ and (x) have?*

Answer. $[x]$ has a simple discontinuity at each integer and is continuous elsewhere. One can prove this formally (if $a \notin \mathbb{Z}$, take $\delta = \min(a-[a], [a]+1-a)$ for any $\epsilon > 0$), but I would be satisfied with a picture. In any case, no additional argument is needed for (x) since any two functions whose sum is continuous will have the same types of discontinuites.

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Theorem. *The definition of uniform continuity is equivalent to the statement: For all $\epsilon > 0$, there exists a $\delta > 0$ such that if diam $E < \delta$ for $E \subset X$, then diam $f(E) < \epsilon$.*

Proof. Assume $f : E \to Y$ is uniformly continuous. Fix $\epsilon > 0$. Choose $\delta > 0$ such that $d(f(x), f(y)) < \frac{\epsilon}{2}$ whenever $d(x, y) < \delta$. Now suppose E is a subset of X having dimaeter less than δ.

Let y_1, y_2 be members of $f(E)$. Write $y_j = f(x_j)$ with $x_1, x_2 \in E$. Since diam$E < \delta$, we have $d(x_1, x_2) < \delta$, whence $d(y_1, y_2) < \frac{\epsilon}{2}$. Thus $\frac{\epsilon}{2}$ is an upper bound of for the set of distances between members of $f(E)$. Since diam $f(E)$ is the *least* upper bound of such distances, we conclude diam$f(E) \leq \frac{\epsilon}{2} < \epsilon$, as desired.

Now assume that the statement concerning diameters is true. Fix $\epsilon > 0$. Choose $\delta > 0$ such that diam $f(E) < \epsilon$ whenever diam $E < \delta$. To complete the proof, we need to show that this δ works in the definition of uniform continuity.

So suppose $x_1, x_2 \in X$ with $d(x_1, x_2) < \delta$. Take $E := \{x_1, x_2\}$. Since diam$E < \delta$, we have $d(f(x_1), f(x_2) = \text{diam} f(E) < \epsilon$ as desired. $\square$

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Theorem. *Let f be a real-valued continuous function defined on a subset E of $\mathbb{R}$.*

(a) If E is bounded and f is uniformly continuous, then f is bounded on E.

(b) *f may fail to be bounded if E is unbounded or if f is not uniformly continuous.*

Proof. Part b) is easiest. Note that counterexamples are required – not an analysis of how the argument for Part a) breaks down. The identity function $i : \mathbb{R} \to \mathbb{R}$ by $i(x) = x$ is uniformly continuous and (its range is) bounded. The reciprocal function $r : (0, 1) \to \mathbb{R}$ by $r(x) = \frac{1}{x}$ has a bounded domain, but its range is again unbounded.

Turning to the proof for Part a), fix $\epsilon = 1$ and apply the uniform continuity hypothesis to find a $\delta > 0$ such that $|f(x) - f(y)| < 1$ whenever $|x - y| < \delta$. Next, express $E = \cup_{j=1}^{n} E_j$ where each E_j has diameter less than δ. Then each image set $f(E_j)$ has diameter less than 1. Thus $f(E)$ is the union of finitely many bounded sets and hence must also be bounded.

{*Comments: In writing up any proof, one must decide on how much detail to provide. The above argument is on the brief side – one could also explain how the E_j's are constructed and/or why finite unions of bounded sets remain bounded. The proof of Problem 9 above, on the other hand, was more fully detailed. Why the difference? First of all, to illustrate both styles, and secondly because the least upper bound concept is so central to this course.*

A totally different approach to the present proof would be to first solve Problem 13. That would provide a continuous extension F of f to the **compact** *domain $\overline{E}$. That would make $f(E)$ a subset of the compact set $F(\overline{E})$.*}

□

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Let E be a bounded subset of a metric space X. For each $x \in X$, define $\rho_E(x) = \inf_{z \in E} d(x, z)$. Then

(a) $x \in \overline{E}$ if and only if $\rho_E(x) = 0$

(b) ρ_E is uniformly continuous.

Proof. Fix $x, y \in X$ and set $S := \{d(x, z) | z \in E\}$. Then 0 is a lower bound for S and $\rho_E(x) = \inf S$ by definition.

For Part (a), first assume $x \in \overline{E}$ and $\epsilon > 0$. Then $N_\epsilon(x) \cap E$ is non-empty which means that ϵ is not a lower bound for S. Thus 0 is in fact the greatest lower bound of S, i.e., $\rho_E(x) = 0$.

Conversely, suppose $\rho_E(x) = 0$ and and let $\epsilon > 0$. Then ϵ is not a lower bound for S, so there is a $z \in E$ with $d(x, z) < \epsilon$, i.e. $z \in N_\epsilon(x) \cap E$. Since this is true for each $\epsilon > 0$, we conclude $x \in \overline{E}$, as desired.

For Part (b), first fix $z \in E$ and note that $\rho_E(y) \leq d(y, z) \leq d(y, x) + d(x, z)$ by definition of inf and the triangle inequality. Transposing, we see that $\rho_E(y) - d(y, x)$ is a lower bound for the set S. Applying the definition of inf once again, we get $\rho_E(y) - d(y, x) \leq \rho_E(x)$. Interchanging the roles of x and y, we see that $|\rho_E(x) - \rho_E(y)| \leq d(x, y)$ for all $x, y \in X$. Thus one can take $\delta = \epsilon$ to establish uniform continuity of ρ_E. □

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Theorem. *Let f be a real function defined on (a, b). Then the set of points at which f has a simple discontinuity is at most countable.*

Proof. There are four sets to consider:

(a) $E := \{x \in (a, b) | f(x-) < f(x+)\}$,

(b) $F := \{x \in (a, b) | f(x-) > f(x+)\}$,

(c) $G := \{x \in (a, b) | f(x-) = f(x+) > f(x)\}$,

(d) $H := \{x \in (a, b) | f(x-) = f(x+) < f(x)\}$.

Associate to each $x \in E$ a triple (p, q, r) of rational numbers satisfying:

(a) $f(x-) < p < f(x+)$,

(b) $a < q < x < r < b$,

(c) $q < t < x$ implies that $f(t) < p$, and

(d) $x < t < r$ implies that $f(t) > p$.

By Theorem 2.13 this set of triples is countable. Many such triples are associated to each $x \in E$. It needs to be shown that no triple can go with two different members of E. Prove this by contradiction. Assume that $x_1 < x_2$ are both assigned to the same triple (p, q, r). The reals are dense in themselves, so there exists an $x_3 \in \mathbb{R}$ such that $x_1 < x_3 < x_2$. We have $q < x_3 < x_2$ so $f(x_3) < p$ by condition (c). On the other hand, we have $x_1 < x_3 < r$ which implies that $f(x_3) > p$ by condition (d). This contradicts trichotomy. Thus x_1 and x_2 cannot be associated to the same triple (p, q, r).

The proof that F is countable is so similar that you should not write out any details. You should, however, at least explain how to associate a triple (p, q, r) of rational numbers with each member of G:

(a) $f(x-) = f(x+) > p > f(x)$,

(b) $a < q < x < r < b$,

(c) $q < t < x$ implies that $f(t) > p$,

(d) $x < t < r$ implies that $f(t) > p$.

There is no need to repeat the countability argument, but it wouldn't hurt to explain why no rational triple can go with two members of G. Again, H deserves no further comment, and the proof is completed by noting that $E \cup F \cup G \cup H$ accounts for simple discontinuites of f. □

11b

Theorem. *Let E be a dense subset of a metric space X, and let f be a uniformly continuous real function defined on E. Then f has a continuous extension from E to X.*

Proof. (outline) Given $x \in X$, find a sequence (x_n) in E converging to x. By Part (a) of this problem, the image sequence $(f(x_n))$ will also be Cauchy and hence will converge to some real number which we will denote $F(x)$. To see that this is a good definition, suppose (y_n) were a another sequence converging to x. Then the "interweaved" sequence $x_1, y_1, x_2, y_2 \ldots$ would also converge to x and hence all subsequences of the image sequence $f(x_1), f(y_1), f(x_2), f(y_2) \ldots$ would have to converge to the same value. In particular, $\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} f(x_n)$ and $F(x)$ is indeed well-defined. When $x \in E$, we can take $x_n = x$ for all n, so $F(x) = f(x)$ and F is indeed an extension of f.

Let $\epsilon > 0$. Apply uniform continuity of f to get $\delta > 0$ so that $|f(p) - f(q)| < \epsilon$ whenever $p, q \in E$ satisfy $d(p, q) < \delta$. To see that this same δ works for the extended function F, let $x, y \in X$ with $d(x, y) < \delta$ and fix sequences $(x_n), (y_n)$ in E which converge to x, y respectively. For each $n \in J$, we have

$$d(x_n, y_n) \le d(x_n, x) + d(x, y) + d(y, y_n), \text{ and}$$

$$|F(x) - F(y)| \le |F(x) - f(x_n)| + |f(x_n) - f(y_n)| + |f(y_n) - F(y)|.$$

Choosing n sufficiently large, we can make the right-hand members of these inequalities less than δ and ϵ respectively which means $|F(x) - F(y)| < \epsilon$ as desired. □

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The "ruler function" $f : \mathbb{R} \to \mathbb{R}$ defined by $f(\frac{m}{n}) := \frac{1}{n}$ when $\frac{m}{n}$ is a fraction in lowest terms, $f(0) := 1$, and $f(x) := 0$ when x is irrational has a simple discontinuity at each rational point and is continuous elsewhere.

Proof. It suffices to show that $\lim_{x\to a} f(x) = 0$ for each real number a. So fix $a \in \mathbb{R}$ and $\epsilon > 0$. Fix an integer $n > \frac{1}{\epsilon}$. Then the interval $(a - \frac{1}{2n!}, a + \frac{1}{2n!})$ has length less than $\frac{1}{n!}$, and hence can contain at most one rational number $p \neq a$ with denominator $\le n$. Take $\delta = |p - a|$ if such a number p exists and $\delta = \frac{1}{2n!}$ otherwise. Then $|f(x)| < \frac{1}{n} < \epsilon$ whenever $0 < |x - a| < \delta$, as desired. □

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Convex functions are continuous. The displayed inequalities compare the slopes of various secant lines associated with a convex function. To establish them algebraically, note that if $s < t < u$, then t is a convex combination of s, u, specifically $t = \lambda s + (1 - \lambda)u$ with $\lambda = \frac{u-t}{u-s}$. Then use the squeeze principle to get the desired continuity result. Full proofs can be found in many textbooks.