1

# #1

This is what we have

$\sup_n E|X_n|^{1+\delta} < \infty$ for some $\delta > 0$.

we need to show

$\{X_n\}_{n\geq 1}$ is u.i

ie to show $\sup_n E\{|X_n| \mathbb{1}_{\{|X_n| \geq \lambda\}}\} \longrightarrow 0$ as $\lambda \longrightarrow \infty$

$$\sup_n E|X_n|^{1+\delta} \geq E|X_n|^{1+\delta} \geq E|X_n|^{1+\delta} \cdot \mathbb{1}_{\{|X_n| \geq \lambda\}}$$

( This is because $|X_n|^{1+\delta} \geq |X_n|^{1+\delta} \cdot \mathbb{1}_{\{|X_n| \geq \lambda\}}$ )

now

$$E|X_n|^{1+\delta} \cdot \mathbb{1}_{\{|X_n| \geq \lambda\}} = E|X_n| |X_n|^{\delta} \cdot \mathbb{1}_{\{|X_n| \geq \lambda\}}$$

$$\geq E|X_n| \cdot \lambda^{\delta} \cdot \mathbb{1}_{\{|X_n| \geq \lambda\}}$$

$$= \lambda^{\delta} \cdot E|X_n| \cdot \mathbb{1}_{\{|X_n| \geq \lambda\}}.$$

$$\sup_n E|X_n|^{1+\delta} \geq \lambda^{\delta} \; E|X_n| \cdot \mathbb{1}_{\{|X_n| \geq \lambda\}}$$

$$\therefore \quad \frac{\sup_n E|X_n|^{1+\delta}}{\lambda^\delta} \geq E|X_n| \cdot \mathbb{1}_{\{|X_n| \geq \lambda\}} \qquad \boxed{\forall n \geq 1}$$

Thus $$\frac{\sup_n E|X_n|^{1+\delta}}{\lambda^\delta} \geq \sup_n E|X_n| \cdot \mathbb{1}_{\{|X_n| \geq \lambda\}}$$

now as $\lambda \to \infty$. $$\frac{\sup_n E|X_n|^{1+\delta}}{\lambda^\delta} \longrightarrow 0$$

$$\therefore \quad \sup_n E|X_n|^{1+\delta} < \infty.$$

$$\therefore \quad \sup_n E|X_n| \cdot \mathbb{1}_{\{|X_n| \geq \lambda\}} \longrightarrow 0 \quad \text{as } n \to \infty.$$

<u>QED</u>

3

#3.

To show.

$$X_n \xrightarrow{r} X.$$

ie

$$E\,|X_n - X|^r \longrightarrow 0.$$

we shall use DCT to show this.

In order to use DCT we need to make sure

(1) $|X_n - X|^r \xrightarrow{P} 0.$

(2) $|X_n - X|^r \leq Y \in L_1.$

Once (1) & (2) are satisfied then by DCT we can say

$$E\,\big|\,|X_n - X|^r - 0\,\big| \longrightarrow 0 \quad \text{as } n \to \infty$$

$$E\,|X_n - X|^r \longrightarrow 0 \quad \text{as } n \to \infty.$$

now

(1) we know

$$X_n \xrightarrow{P} X$$

$$\Rightarrow \quad |X_n - X| \xrightarrow{P} 0$$

$$\Rightarrow \quad |X_n - X|^r \xrightarrow{P} 0. \quad \Leftrightarrow \quad P(|X_n - X|^r > \varepsilon) \to 0$$

$$P(|X_n - X| > \varepsilon^{\frac{1}{r}}) \to 0$$

$$\therefore \quad P(|X_n - X| > \varepsilon') \to 0$$

Thus (1) is true.

$$\boxed{\varepsilon' = \varepsilon^{\frac{1}{r}}}$$

now

(2). we know

$$|X_n - X|^r \le C_r\{|X_n|^r + |X|^r\}$$

$|x+y|^r \le C_r\{|x|^r + |y|^r\}$

now. from the problem itself we know

$$|X_n| \le C$$

now $\quad |X| \le |X_n - X| + |X_n|.$

$$\therefore \quad P(|X| > c+d) \le P(|X_n - X| > d) + P(|X_n| > c)$$

now $\quad P(|X_n| > c) = 0.$

$$\therefore \quad P(|X| > c+d) \le P(|X_n - X| > d)$$

now

$$P(|X| > c+d) \le P(|X_n - X| > d)$$

$$\Rightarrow \quad P(|X| > c+d) \le \lim_n P(|X_n - X| > d).$$

since $X_n \xrightarrow{P} X$

$$\therefore \quad \lim_n P(|X_n - X| > d) = 0$$

$$\Rightarrow \quad P(|X| > c+d) \le 0$$

$$\Rightarrow \quad P(|X| \le c+d) = 1.$$

So we have

$$|X_n| \le c \quad \forall n \geq 1$$

and $|X| \le c+d$.

now $|X_n - X|^r \le C_r\{|X_n|^r + |X|^r\}$

$$\Rightarrow \quad |X_n - X|^r \le C_r\{|X_n|^r + |X|^r\} \le C_r\{c^r + (c+d)^r\}.$$

now our $Y = C_r\{c^r + (c+d)^r\}$

because.

$$\int_\Omega |Y| \, dP = C_r \{ c^r + (c+d)^r \} \cdot P(\Omega)$$
$$= C_r \{ c^r + (c+d)^r \} < \infty.$$

$\Rightarrow \quad Y \in L_1.$

$\Rightarrow$ (2) is satisfied.

ie $\quad |X_n - X|^r \leq Y \in L_1$

$\Rightarrow$ Thus since (1) & (2) are satisfied we use.

DCT to ~~show~~ state

$E|X_n - X|^r \longrightarrow 0$

QED

7

#2.

1 $X_n \xrightarrow{P} X.$

ie. $P(|X_n - X| > \varepsilon) \longrightarrow 0$ for any $\varepsilon > 0$

as $n \to \infty$.

~~to~~ since $X = 0$. enough to show

$$P(|X_n| > \varepsilon) \longrightarrow 0 \quad \text{as } n \to \infty.$$

now

$$P(|X_n| > \varepsilon) = P(X_n \neq 0) = P(X_n = n^2) = \frac{1}{n^4}.$$

now $\frac{1}{n^4} \to 0$ as $n \to \infty$.

$\therefore$ $P(|X_n| > \varepsilon) \longrightarrow 0$ as $n \to \infty$.

$\therefore$ $X_n \xrightarrow{P} X.$

3

now. ~~$X_n$~~ Let $r = 2$.

$\therefore$ $E|X_n - X|^2 = E|X_n|^2 \quad \because X = 0.$

$= 1$

Thus $E|X_n - X|^2 \not\longrightarrow 0 \quad (\because E|X_n - X|^2 = 1 \ \forall n \geq 1)$

Thus

$$X_n \overset{2}{\not\longrightarrow} X.$$

2

$$X_n \xrightarrow{as} X.$$

enough to show.

for any $k = 1, 2, 3, 4, \dots$

$$P\left(\limsup |X_n - X| > \frac{1}{k}\right) = 0$$

now $\parallel$

is $P\left(\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} |X_m - X| > \frac{1}{k}\right)$

$$= \lim_{n \to \infty} P\left(\bigcup_{m=n}^{\infty} |X_m - X| > \frac{1}{k}\right)$$

$\left( \because \; P\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim P(A_n) \text{ when } A_1 \subset A_2 \subset A_3 \subset \cdots \right)$

$$\leq \lim_{n \to \infty} \sum_{m=n}^{\infty} P\left(|X_m - X| > \frac{1}{k}\right).$$

$$= \lim_{n \to \infty} \sum_{m=n}^{\infty} P(|X_m| \neq 0)$$

$\because \left[ P\left(|X_m - X| > \frac{1}{k}\right) = P(|X_m| \neq 0) \text{ as } X = 0, \text{ and } k > 0 \right.$

$$= \lim_{n\to\infty} \sum_{m=n}^{\infty} \frac{1}{m^4}$$

$$= 0$$

$$\because \sum_{m=1}^{\infty} \frac{1}{m^4} < \infty.$$

Thus,

$$P\left( \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} |X_m - X| > \frac{1}{k} \right) = 0.$$

Thus $P\left( \limsup |X_m - X| > \frac{1}{k} \right) = 0.$

for any $k = 1, 2, 3, \ldots$

Thus $P(X_n \not\to X) = 0$

$P(X_n \to X) = 1.$