# Homework 2 – STA 5446
## Fall 2005

1. Reading exercise: Page 20, PfS.

Must know

2. Prove that simple and elementary functions are measurable. (Claim (7), page 25, Proposition 2.2.3 in PfS).

Try Exam

3. Reading exercise: Definitions 1.1.5 and 1.1.6 page 8 in PfS.

4. Find $\overline{\lim} A_n$ and $\underline{\lim} A_n$, where

$$A_n = \begin{cases} \left(\frac{1}{n}, \frac{2}{3} - \frac{1}{n}\right), & \text{for } n = 1, 3, 5, \ldots \\ \left(\frac{1}{3} - \frac{1}{n}, 1 + \frac{1}{n}\right), & \text{if } n = 2, 4, 6, \ldots \end{cases}$$

$\mathcal{B} = \sigma(\mathcal{C}_1) = \sigma(\mathcal{C}_2) = \sigma(\mathcal{C}_3) \ldots$

$1 \Rightarrow 3$ Any open set in $\mathbb{R}$ can be expressed as a countable union of open intervals.

$\mathcal{C}_1$ = class of all open sets

$\mathcal{C}_2 = \{[a, \infty) : a \in \mathbb{R}\}$

$2 \Rightarrow 3$ $\bigcap_{n=1}^{\infty} (a + 1/n, \infty) = [a, \infty)$

#5 (b) $\overline{\lim}\ A_n = \left( \bigcap_{n=1}^{\infty} \bigcup_{m \geq n} A_m \right)$

$$\left(\overline{\lim}\ A_n\right)^c = \left( \qquad \right)^c$$

$$= \bigcup_{n=1}^{\infty} \bigcap_{m \geq n} (A_m^c)$$

$$= \underline{\lim}\ A_n^c$$

$$\therefore \left[\mu\left(\underline{\lim}\ A_n^c\right)\right]^c = \mu\left(\overline{\lim}\ A_n\right)$$

& use $\mu(A^c) = \mu(\Omega) - \mu(A)$

$$\underline{\lim}\ \mu(A^c) = \underline{\lim}\ \mu(\Omega) - \underline{\lim}\ \mu(A)$$

5. Show that (a) $\mu(\underline{\lim}\, A_n) \leq \liminf \mu(A_n)$ #

Proof Not need

and

(b) $\limsup \mu(A_n) \leq \mu(\overline{\lim}\, A_n)$.

( Exercise 1.1.2 page 9 ). where $\mu(\Omega) < \infty$

Use $\mu(A^c) = \mu(\Omega) - \mu(A)$ & $\overline{\lim} = \bigcap_{n=1}^{\infty} \bigcup_{m \geq n} A_m$

• 6. Let $\Omega$ and $\tilde{\Omega}$ be arbitrary sets and let $X : \tilde{\Omega} \to \Omega$ be any function. Show that if $\mathcal{F}$ is a $\sigma$-field on $\Omega$, then $\tilde{\mathcal{F}} \equiv \{ X^{-1}(A), A \in \mathcal{F} \}$ is a $\sigma$-field on $\tilde{\Omega}$.

Know meaning

• 7. Let $\Omega$ and $\tilde{\Omega}$ be arbitrary sets and let $X : \tilde{\Omega} \to \Omega$ be any function. Show that if $\tilde{\mathcal{F}}$ is a $\sigma$-field on $\tilde{\Omega}$, then $\mathcal{F} = \{ A \subset \Omega, X^{-1}(A) \in \tilde{\mathcal{F}} \}$ is a $\sigma$-field on $\Omega$.

Know meaning

$$X : \underset{\tilde{\mathcal{F}}}{\tilde{\Omega}} \longrightarrow \underset{\mathcal{F}}{\Omega}$$

{ set of all preimages of A where $A \in \sigma$-field in $\Omega$ } is a $\sigma$-field in $\tilde{\Omega}$

{ Set of all $A \in \Omega$ whose preimages belong to some $\sigma$-field on $\tilde{\Omega}$ } is also a $\sigma$-field in $\Omega$

8. Find an example of a function

★ $X: \tilde{\Omega} \to \Omega$ and a $\sigma$-field $\tilde{\mathcal{F}}$ on $\tilde{\Omega}$ such that

$\Omega = \{1,2,3\}$
& use counter example

$X(\tilde{\mathcal{F}}) = \{X(A) : A \in \tilde{\mathcal{F}}\}$ is NOT a $\sigma$-field.

Ask Shuva

9. Exercise 2.2.1 page 28 PfS.

Must know

10. a) Let $\lambda$ be the Lebesgue measure.
Find $\lambda(\{x : |x-n| < \frac{1}{2^n}$ for some $n \in \mathbb{N}\})$.

b) Recall that the Dirac measure is

$$\delta_{\omega_0}(A) = \begin{cases} 1, & \omega_0 \in A \\ 0, & \text{otherwise} \end{cases},$$

for a fixed element $\omega$ in $\Omega$.
Note that we can define $\delta_\omega$ on $2^\Omega$.
Show that $\delta_\omega$ is a probability measure.
What is the largest set of measure zero?
What is the largest set of measure 1?

- $\sigma$- field
- Examples of measures
- Defined L-S measure $\mu$
- $\mu \Longleftrightarrow F$ was established
- The Lebesgue measure was defined $(F(x) = x)$ as a case of L-S meas
- The Lebesgue measure cannot be extended to $2^{\mathbb{R}}$
- But there are measures (L-S) which can be defined on all $2^{\mathbb{R}}$ eg: the dirac measure
- can the L-S measure be characterized to these which cannot be defined on $2^{\mathbb{R}}$

Theorem say the measure on $2^{\mathbb{R}}$ cannot be both.

~~st~~ ① assign finite values to finite intervals

② also gives measure 1 to all singletons