#1] Ω is the set of all possible outcomes and we call it the sample space.
The measurable subsets A in the collection $\mathcal{A}$ are reffered to as events
P is called a probablity measure provided.

$P(\Omega) = 1$ and P is a measure.

An $\uparrow$ rt continuous function F on R with $F(-\infty) = 0$ & $F(\infty) = 1$ is called a dist^n function

Correspondence theorem : $P((a,b]) \equiv F(b) - F(a)$

#2] Prove that simple elementary functions are measurable.

(i) $f : (\Omega, \mathcal{A}) \longrightarrow (\mathbb{R}, \mathcal{B})$

$$f(x) = \sum_{i=1}^{n} a_i I_{A_i}(x) \quad \text{where } A_i \in \mathcal{A} \text{ and } \Omega = \sum_{i=1}^{n} A_i$$

Now f can take finitely many values (n values $a_1, \ldots a_n$)
let $c_1 < c_2 < c_3 \ldots < c_M$ be the values that f takes.

let $\mathcal{B} = \sigma[\mathcal{C}_2]$ where $\mathcal{C}_2 = \{[a,\infty) : a \in \mathbb{R}\}$

To show f is measurable it is enough to show

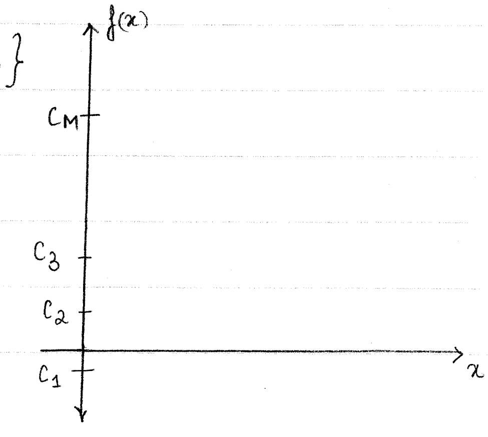
$f^{-1}(\mathcal{C}_2) \subset \mathcal{A}$ ie $f^{-1}(A) \in \mathcal{A} \quad \forall A \in \mathcal{C}_2$

ie $f^{-1}([a,\infty)) \in \mathcal{A} \quad \forall a \in \mathbb{R}$

If $a \leq c_1 \Rightarrow f^{-1}([a,\infty)) = \{x : f(x) \in [a,\infty)\}$

$= \Omega$

$\because$ all values of $f(x) \geq c_1 \geq a$ for any $x \in \Omega$



$A_1 \cdots A_n$

If $c_1 < a \leq c_2 \Rightarrow f^{-1}([a,\infty)) = \Omega \setminus A_{n_0}$ for some $n_0 \in \{1, 2 \cdots n\}$

$= A_{n_0}^c$

If $c_2 < a \leq c_3 \Rightarrow f^{-1}([a,\infty)) = \Omega \setminus (A_{n_0} \cup A_{n_1}) = (A_{n_0} \cup A_{n_1})^c$

$\vdots$

If $a > c_M \Rightarrow f^{-1}([a,\infty)) = (A_{n_0} \cup A_{n_1} \cdots \cup A_{n_M})^c = \phi$

Now $A_{n_0}, A_{n_1}, \cdots A_{n_M} \in \mathcal{A}$ and $\mathcal{A}$ is a σ-field

$\Rightarrow \phi$, all countable unions, complements, $\Omega \in \mathcal{A}$

$\therefore f^{-1}([a,\infty)) \in \mathcal{A} \quad \forall a \in \mathbb{R}$

$\therefore f^{-1}(\mathcal{C}_2) \subset \mathcal{A} \Rightarrow f$ is measurable

Claim : any elementary funct^n can be expressed as a limit of seq of simple functions.

ie $g(x) = \lim_{n\to\infty} f_n(x)$ where $f_n(x) = \sum_{i=1}^{n} a_i I_{A_i}(x)$

Claim limits of a measurable funct^n are also measurable

$\therefore g(x)$ is also measurable ie any elementary funct^n is also measurable.

#3] $\underline{\lim} A_n = \bigcup_{n=1}^{\infty} \bigcap_{m\geq n}^{\infty} A_m$ & $\overline{\lim} A_n = \bigcap_{n=1}^{\infty} \bigcup_{m\geq n}^{\infty} A_m$

$\underline{\lim} A_n \subset \overline{\lim} A_n$

For a seq of real #'s a_n

$\liminf a_n = \lim_{n\to\infty} \left(\inf_{k\geq n} a_k \right)$ & $\limsup a_n = \lim_{n\to\infty} \left(\sup_{k\geq n} a_k \right)$

To show $\underline{\lim}\, A_n \subset \overline{\lim}\, A_n$

$$\omega \in \underline{\lim}\, A_n \Rightarrow \omega \in \bigcup_{n=1}^{\infty} \bigcap_{m \geq n} A_m$$

$$\Rightarrow \omega \in \bigcap_{m \geq n_0} A_m \quad \text{for some } n_0.$$

$$\Rightarrow \omega \in A_m \quad \forall\ m \geq n_0$$

$\Rightarrow \omega \in A_n \ \forall\ n$ but finitely many of them

Now if $\omega \in \overline{\lim}\, A_n \Rightarrow \omega \in \bigcap_{n=1}^{\infty} \bigcup_{m \geq n} A_m$

$$\Rightarrow \omega \in \bigcup_{m \geq n} A_m \quad \forall\ n$$

$\Rightarrow$ given any m $\exists$ some n_0 st $\omega \in A_{n_0}$

$\Rightarrow \omega \in$ infinitely many A_n's

$$\therefore \underline{\lim}\, A_n \subset \overline{\lim}\, A_n$$

#4]

$$A_n = \left(\tfrac{1}{n}, \tfrac{2}{3} - \tfrac{1}{n}\right) \qquad n = 5, 7, 9 \ldots$$

$$= \left(\tfrac{1}{3} - \tfrac{1}{n}, 1 + \tfrac{1}{n}\right) \qquad n = 2, 4, 6, 8 \ldots$$

let $A_1 = \mathbb{R}$ and $A_3 = \mathbb{R}$ { altering finitely many sets will not chang the limits

$$\underline{\lim}\, A_n = \bigcup_{n=1}^{\infty} \bigcap_{m \geq n} A_m$$

$$= \bigcup_{n=1}^{\infty} \left[\left(\bigcap_{\substack{m \geq n \\ m:\text{odd}}} A_m \right) \cap \left(\bigcap_{\substack{m \geq n \\ m:\text{even}}} A_m \right) \right]$$

Now $\bigcap_{\substack{m \geq n \\ m:\text{odd}}}^{\infty} A_m = \underbrace{\bigcap_{\substack{m \geq n \\ m:\text{odd}}}^{\infty} \left(\tfrac{1}{m}, \tfrac{2}{3} - \tfrac{1}{m}\right)}_{\text{decreasing seq of sets}} = \left(\tfrac{1}{n}, \tfrac{2}{3} - \tfrac{1}{n}\right)$ {n: odd

$$\bigcap_{\substack{m \geq n \\ m:\text{even}}} A_m = \bigcap_{\substack{m \geq n \\ m:\text{even}}} \overbrace{\left(\tfrac{1}{3} - \tfrac{1}{m}, 1 + \tfrac{1}{m}\right)}^{\downarrow \text{ seq of sets}} = \left[\tfrac{1}{3}, 1\right]$$

$$\therefore \bigcap_{m \geq n} A_m = \left(\tfrac{1}{n}, \tfrac{2}{3} - \tfrac{1}{n}\right) \cap \left[\tfrac{1}{3}, 1\right] = \left[\tfrac{1}{3}, \tfrac{2}{3} - \tfrac{1}{n}\right)$$

$$\underline{\lim} \; A_n = \bigcup_{n=1}^{\infty} \left[\tfrac{1}{3}, \tfrac{2}{3} - \tfrac{1}{n}\right) = \left[\tfrac{1}{3}, \tfrac{2}{3}\right)$$

$$\overline{\lim} \; A_n = \bigcap_{n=1}^{\infty} \bigcup_{m \geq n}^{\infty} A_m$$

$$= \bigcap_{n=1}^{\infty} \left[\left(\bigcup_{\substack{m \geq n \\ m:\text{even}}}^{\infty} A_m \right) \cup \left(\bigcup_{\substack{m \geq n \\ m:\text{odd}}}^{\infty} A_m \right) \right]$$

Now

$$\bigcup_{\substack{m \geq n \\ m:\text{even}}}^{\infty} A_m = \bigcup_{\substack{m \geq n \\ m:\text{even}}} \underbrace{\left(\tfrac{1}{3} - \tfrac{1}{m}, 1 + \tfrac{1}{m}\right)}_{\text{decreasing seq}} = \left(\tfrac{1}{3} - \tfrac{1}{n}, 1 + \tfrac{1}{n}\right) \quad \{n \geq 2, 4 \ldots$$

$$\bigcup_{\substack{m \geq n \\ m:\text{odd}}} A_m = \bigcup_{\substack{m \geq n \\ m:\text{odd}}}^{\infty} \underbrace{\left(\tfrac{1}{m}, \tfrac{2}{3} - \tfrac{1}{m}\right)}_{\uparrow \text{ seq of sets}} = \left(0, \tfrac{2}{3}\right)$$

$$\therefore \overline{\lim} \; A_n = \bigcap_{n=1}^{\infty} \left[\left(0, \tfrac{2}{3}\right) \cup \left(\tfrac{1}{3} - \tfrac{1}{n}, 1 + \tfrac{1}{n}\right) \right] \quad \{ n \text{ even}$$

$$= \bigcap_{n=1}^{\infty} \underbrace{\left(0, 1 + \tfrac{1}{n}\right)}_{\downarrow \text{ seq of sets}}$$

$$= (0, 1]$$

$$\left\{\begin{array}{lll} a_n \leq b_m & \forall \; m \geq n \\ a_n \leq \inf\limits_{m \geq n} b_m & \end{array}\right\} \text{—} (**)$$

#5 (a) Show $\mu(\underline{\lim} \; A_n) \leq \underline{\lim} \; \mu(A_n)$

$$\mu(\underline{\lim} \; A_n) = \mu\left(\bigcup_{n=1}^{\infty}\left(\bigcap_{m \geq n}^{\infty} A_m\right)\right)$$

$$\cancel{\leq \sum_{n=1}^{\infty} \mu\left(\bigcap_{m \geq n}^{\infty} A_m\right)} \quad \{\text{countable subadditivity}$$

Now let $B_n \equiv \bigcap\limits_{m \geq n}^{\infty} A_m \Rightarrow B_n$ is an $\uparrow$ seq

$$\underline{\lim} \; A_n = \bigcup_{n=1}^{\infty} B_n = \lim_{n \to \infty} B_n$$

$$\mu(\underline{\lim} \; A_n) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right)$$

$$= \lim_{n \to \infty} \mu(B_n) \quad \{\text{Monotone prop of meas} \text{ —— } (*)$$

Assumes $\mu(\Omega) < \infty$ and all $\{B_n\} \uparrow$ seq so $\{\mu(B_n)\} \uparrow$ seq and has limit

$$\text{Now} \quad \mu(B_n) = \mu\left(\bigcap_{m \geq n}^{\infty} A_m\right) \leq \mu(A_n) \quad \left\{ \bigcap_{m \geq n} A_m \subset A_n \right.$$

$$\therefore \underline{\lim} \; \mu(B_n) \leq \underline{\lim} \; \mu(A_n)$$

$$\therefore \lim_{n \to \infty} \mu(B_n) \leq \underline{\lim} \; \mu(A_n)$$

$$\therefore \mu(\underline{\lim} \; A_n) \leq \underline{\lim} \; \mu(A_n) \quad \{\text{from } (*)$$

$$\mu(\underline{\lim} \; A_n) = \lim_{n \to \infty}\left[\mu(B_n)\right] \leq \lim_{n \to \infty}\left[\inf_{m \geq n} \mu(A_m)\right] \quad \text{using } (**)$$

$$\mu(\underline{\lim} \; A_n) \leq \underline{\lim} \; \mu(A_n) \quad \text{by definition}$$

5(b)] If $\mu(\Omega) < \infty$ then $\mu(\overline{\lim} A_n) \geq \overline{\lim}\, \mu(A_n)$

Consider $\mu(\overline{\lim} A_n) = \mu\left(\bigcap_{n=1}^{\infty} \bigcup_{m\geq n}^{\infty} A_m\right)$

$$= \mu\left[\left(\bigcup_{n=1}^{\infty} \bigcap_{m\geq n}^{\infty} A_m^c\right)^c\right]$$

$$= \mu(\Omega) - \mu\left(\bigcup_{n=1}^{\infty} \bigcap_{m\geq n}^{\infty} A_m^c\right)$$

$$= \mu(\Omega) - \mu\left(\underline{\lim}\, A_n^c\right) \qquad \text{—}(*)$$

Now $\mu(\underline{\lim}\, A_n^c) \leq \underline{\lim}\, \mu(A_n^c)$ from 5(a)

$$\therefore \mu(\Omega) - \mu(\underline{\lim}\, A_n^c) \geq \mu(\Omega) - \underline{\lim}\, \mu(A_n^c)$$

$$\mu(\Omega) - \mu(\underline{\lim}\, A_n^c) \geq \mu(\Omega) - \underline{\lim}\, [\mu(\Omega) - \mu(A_n)]$$

$$\mu(\Omega) - \mu(\underline{\lim}\, A_n^c) \geq \mu(\Omega) - \mu(\Omega) - \underline{\lim}\,(-\mu(A_n))$$

$$\therefore \mu(\overline{\lim}\, A_n) \geq -\underline{\lim}\,(-\mu(A_n))$$

$$\geq \overline{\lim}\, [\mu(A_n)]$$

$\Rightarrow$ $\left\{ \text{since } \underline{\lim}\,(a_n) = -\overline{\lim}\,(-a_n) \right\}$

#6] $X : \tilde{\Omega} \longrightarrow \Omega$ is an arbitrary function.

$\mathcal{F}$ is a σ-field on Ω

$\tilde{\mathcal{F}} \equiv \{ X^{-1}(A) \ : \ A \in \mathcal{F} \}$

To show $\tilde{\mathcal{F}}$ is a σ-field on $\tilde{\Omega}$

(i) $\tilde{\Omega} \in \tilde{\mathcal{F}}$

(ii) $A \in \tilde{\mathcal{F}} \Rightarrow A^c \in \tilde{\mathcal{F}}$

(iii) $A_1, A_2 \cdots \in \tilde{\mathcal{F}} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \tilde{\mathcal{F}}$

(i) $\tilde{\Omega} = X^{-1}(\Omega)$ & $\Omega \in \mathcal{F}$ {since $\mathcal{F}$ is a σ-field on Ω}

$\therefore$ $\tilde{\Omega} \in \tilde{\mathcal{F}}$ {by definition of $\tilde{\mathcal{F}}$}

(ii) $A \in \tilde{\mathcal{F}} \Rightarrow A = X^{-1}(B)$ where $B \in \mathcal{F}$

Now $B^c \in \mathcal{F}$ also $\Rightarrow X^{-1}(B^c) = (X^{-1}(B))^c \in \tilde{\mathcal{F}}$

$\Rightarrow X^{-1}(B^c) = A^c \in \tilde{\mathcal{F}}$

(iii) $A_1, A_2 \ldots \in \mathcal{F} \Rightarrow A_1 = X^{-1}(B_1)$ where $B_1 \in \mathcal{F}$

$A_2 = X^{-1}(B_2)$ where $B_2 \in \mathcal{F}$

$\vdots$

Since $B_1, B_2 \ldots \in \mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} B_n \in \mathcal{F}$ ($\mathcal{F}$ is σ-field)

$\therefore$ $X^{-1}\left(\bigcup_{n=1}^{\infty} B_n\right) \in \tilde{\mathcal{F}}$ {using def of $\tilde{\mathcal{F}}$

$\Rightarrow \bigcup_{n=1}^{\infty} X^{-1}(B_n) = \bigcup_{n=1}^{\infty} A_n \in \tilde{\mathcal{F}}$ $\left\{ \bigcup_{n=1}^{\infty} X^{-1}(B_n) = X^{-1}\left(\bigcup_{n=1}^{\infty} B_n\right) \right.$

$\therefore$ $\tilde{\mathcal{F}}$ is a σ-field on $\tilde{\Omega}$

#7 $X : \tilde{\Omega} \to \Omega$ is any function

$\tilde{\mathcal{F}}$ is a σ-field on $\tilde{\Omega}$

$\mathcal{F} \equiv \{ A \subset \Omega \, , \, X^{-1}(A) \in \tilde{\mathcal{F}} \}$

To show $\mathcal{F}$ is a σ-field on Ω (i) $\Omega \in \mathcal{F}$

(ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

(iii) $A_1, A_2 \ldots \in \mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$

(i) $\Omega \subset \Omega$ and $X^{-1}(\Omega) = \tilde{\Omega} \in \tilde{\mathcal{F}}$ $\{ \because \tilde{\mathcal{F}}$ is σ-field on $\tilde{\Omega}$

$\Rightarrow \Omega \in \mathcal{F}$

(ii) $A \in \mathcal{F} \Rightarrow A \subset \Omega$ and $X^{-1}(A) \in \tilde{\mathcal{F}}$

Now $A^c \subset \Omega$ and $X^{-1}(A^c) = (X^{-1}(A))^c$

But $X^{-1}(A) \in \tilde{\mathcal{F}} \Rightarrow (X^{-1}(A))^c \in \tilde{\mathcal{F}}$ also

$\therefore$ $A^c \subset \Omega$ and $X^{-1}(A^c) \in \tilde{\mathcal{F}}$

$\Rightarrow A^c \in \mathcal{F}$

(iii) $A_1, A_2 \ldots \in \mathcal{F} \Rightarrow A_1 \subset \Omega, A_2 \subset \Omega, \ldots$

and $X^{-1}(A_1), X^{-1}(A_2) \ldots \in \tilde{\mathcal{F}}$

$\Rightarrow \bigcup_{n=1}^{\infty} X^{-1}(A_n) \in \tilde{\mathcal{F}}$

$\Rightarrow X^{-1}\left(\bigcup_{n=1}^{\infty} A_n\right) \in \tilde{\mathcal{F}}$ —(*)

Also $A_1, A_2 \ldots \subset \Omega \Rightarrow \bigcup_{n=1}^{\infty} A_n \subset \Omega$ —(**)

$\therefore \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$ $\{$(*) & (**) & def of $\mathcal{F}$

$\therefore$ $\mathcal{F}$ is a σ-field on Ω

$X : \tilde{\Omega} \to \Omega$ must be defined pt-wise ie $X(\omega)$ and then it follows
$X(A) = \{X(\omega_1), X(\omega_2)\}$ if $A = \{\omega_1, \omega_2\}$

#8] $X : \tilde{\Omega} \to \Omega$

$\tilde{\mathcal{F}}$ is a σ-field on $\tilde{\Omega}$

$\mathcal{F} = \{X(A) : A \in \tilde{\mathcal{F}}\}$

Let $\tilde{\Omega} = \{-2, -1, 1, 2\}$

$\Omega = \{1, 4\}$

$f(\omega) = \omega^2$ is my funct^{n} $f : \tilde{\Omega} \to \Omega$

$\tilde{\mathcal{F}} = \sigma(\{1\}) = \{\phi, \{1\}, \{-2, -1, 2\}, \tilde{\Omega}\}$

$\mathcal{F} = f(\tilde{\mathcal{F}}) = \{\phi, \{1\}, \{1, 4\}\}$

But $\{1\}^c = \{4\} \notin \mathcal{F} \Rightarrow \mathcal{F}$ is not a σ-field on Ω

#9] $(\Omega, \mathcal{A}) = (\mathbb{R}_2, \mathcal{B}_2)$ where $\mathcal{B}_2$ is the σ-field generated by open subsets of $\mathbb{R}_2$

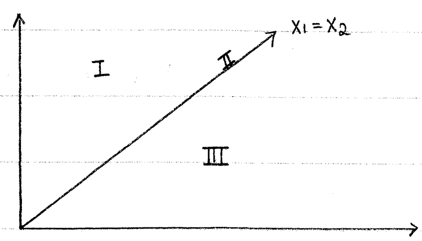
This σ-field contains all sets $B \times R$ and $R \times B$ where $B \in \mathcal{B}$

$B_1 \times B_2 = \{(r_1, r_2) : r_1 \in B_1, r_2 \in B_2\}$

Define $Z_1 \equiv \sqrt{X_1^2 + X_2^2}$ & $Z_2 = \text{sign}(X_1 - X_2)$

Give geometric descriptions of the σ-fields $\mathcal{F}(Z_1), \mathcal{F}(Z_2), \mathcal{F}(Z_1, Z_2)$

$$Z_2 = \begin{cases} 1 & \text{if } x_1 > x_2 \\ 0 & x_1 = x_2 \\ -1 & x_1 < x_2 \end{cases}$$



$Z_2 : (X_1, X_2) \longrightarrow (\mathbb{R}, \mathcal{B})$

$\mathcal{F}(Z_2) = \{Z_2^{-1}(A) : A \in \mathcal{B}\}$

$$Z_2^{-1}(A) = \begin{cases} \phi & \text{if } 0, 1, -1 \notin A \\ \{I\} \cup \{II\} & \text{if } 0 \text{ OR } 1 \in A \\ \{II\} \cup \{III\} & \text{if } 0 \text{ OR } 1 \in A \end{cases}$$

$$Z^{-1}(A) = \begin{cases} \{I\} \cup \{III\} & \text{if } -1, 1 \in A \\ \{I\} \cup \{II\} \cup \{III\} & \text{if } 1, 0, -1 \in A \end{cases}$$

$\mathcal{F}(Z_2) = \{$ set of finite unions of $\{I\}, \{II\}, \{III\}$ with ϕ included

$Z_1 = \sqrt{X_1^2 + X_2^2} \quad : (X_1, X_2) \longrightarrow (\mathbb{R}, \mathcal{B})$

$\mathcal{F}(Z_1) = \{ Z_1^{-1}(A) \; : A \in \mathcal{B} \}$

$= \sigma$ (generated by $Z_1^{-1}(\mathcal{C})$) where $\mathcal{B} = \sigma[\mathcal{C}]$

$= \sigma$ (generated by $Z_1^{-1}((-\infty, a])$) ; $\mathcal{B} = \sigma[(-\infty, a]]$

$= \sigma \{ (x_1, x_2) : Z_1(X_1, X_2) \leq a \}$

$= \sigma \{ (x_1, x_2) : \sqrt{x_1^2 + x_2^2} \leq a \}$

$= \sigma \{$ generated by disc with radius 'a' $\}$

$\mathcal{F}(Z_1, Z_2) = \sigma \{$ generated by $\{I\}, \{II\}, \{III\}$ and discs of radius 'a' $\}$

#10] (a) λ is Lebesgue measure

$$\left.\begin{aligned} A &= \{x : |x-n| < 1/2^n \text{ for some } n \in \mathbb{N}\} \\ &= \bigcup_{n=1}^{\infty} \{x : |x-n| < 1/2^n\} \end{aligned}\right\} \text{---}(*)$$

$$\lambda(A) = \lambda\left(\bigcup_{n=1}^{\infty} \{x : |x-n| < 1/2^n\}\right)$$

$$= \sum_{n=1}^{\infty} \lambda(\{x : |x-n| < 1/2^n\}) \qquad \{\because \text{disjoint union}$$

$$= \sum_{n=1}^{\infty} \left[\left(n+\frac{1}{2^n}\right) - \left(n-\frac{1}{2^n}\right)\right]$$

$$= \sum_{n=1}^{\infty} 2/2^n \qquad = 2$$

Proof of (*)

let $A = \{x : |x-n| < 1/2^n \text{ for some } n \in \mathbb{N}\}$

$B = \bigcup_{n=1}^{\infty} \{x : |x-n| < 1/2^n\}$

$$x_0 \in A \Rightarrow |x_0 - n_0| < 1/2^{n_0} \text{ for some } n_0 \in \mathbb{N}$$

$$\Rightarrow x_0 \in \bigcup_{n=1}^{\infty} \{x : |x-n| < 1/2^n\} = B$$

$\therefore A \subseteq B$

Now let $x_0 \in B \Rightarrow x_0 \in \bigcup_{n=1}^{\infty} \{x : |x-n| < 1/2^n\}$

$$\Rightarrow |x_0 - m| < 1/2^m \text{ for some } m \in \mathbb{N}$$

$$\Rightarrow x_0 \in A$$

$\therefore B \subseteq A$

$\therefore A = B$

10] (b) $\delta_{\omega_0}(A) = \begin{cases} 1 & \omega_0 \in A \\ 0 & \text{ow} \end{cases}$ for $\omega_0 \in \Omega$ & ω_0 fixed

$\delta_{\omega_0}(\omega) \geq 0$ for all $\omega \in \Omega$

$\delta_{\omega_0}(\phi) = 0$ since $\omega_0 \notin \phi$.

$\delta_{\omega_0}(\Omega) = 1$ since $\omega_0 \in \Omega$

To show $\delta_{\omega_0}(\sum A_i) = \sum \delta_{\omega_0}(A_i)$

$\delta_{\omega_0}\left(\sum_{i=1}^{\infty} A_i\right) = 0$ if $\omega_0 \notin \sum_{i=1}^{\infty} A_i$

But if $\omega_0 \notin \left(\sum_{i=1}^{\infty} A_i\right) \Rightarrow \omega_0 \notin A_i \quad \forall i$

$\Rightarrow \delta_{\omega_0}(A_i) = 0 \quad \forall i$

$\Rightarrow \sum_{i=1}^{\infty} \delta_{\omega_0}(A_i) = 0$

Also $\delta_{\omega_0}\left(\sum_{i=1}^{\infty} A_i\right) = 1$ if $\omega \in \sum_{i=1}^{\infty} A_i$

But if $\omega_0 \in \sum_{i=1}^{\infty} A_i \Rightarrow \exists\, n_0$ st $\omega_0 \in A_{n_0}$ & $\omega_0 \notin A_i \; \forall\, i \neq n_0$

$\Rightarrow \delta_{\omega_0}(A_{n_0}) = 1$ & $\delta_{\omega_0}(A_i) = 0 \quad \forall\, i \neq n_0$

$\Rightarrow \sum_{i=1}^{\infty} \delta_{\omega_0}(A_i) = \delta_{\omega_0}(A_{n_0}) + \sum_{i \neq n_0} \delta_{\omega_0}(A_i)$

$= 1$

$A = \Omega \setminus \{\omega_0\}$ is the largest set of measure zero

$B = \{\omega_0\}$ is the smallest set of measure one

$C = \Omega$ is the largest set of measure one