① pg

HOMEWORK # 2

For any set $(X^{-1}A)^c \equiv X^{-1}A^c$

$$\bigcup_{n=1}^{\infty}(X^{-1}A_n) \equiv X^{-1}\left(\bigcup_{n=1}^{\infty}A_n\right)$$

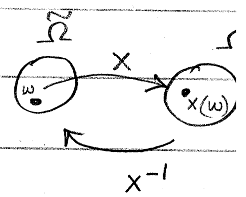
6) $X : \tilde{\Omega} \to \Omega$, is any funct^n & $\mathcal{F}$ is a σ-field on Ω

Show $\tilde{\mathcal{F}} \equiv \{X^{-1}(A),\ A \in \mathcal{F}\}$ is a σ-field on $\tilde{\Omega}$

Check : (i) $\tilde{\Omega} \in \tilde{\mathcal{F}}$

(ii) $A \in \tilde{\mathcal{F}} \Rightarrow A^c \in \tilde{\mathcal{F}}$

(iii) $A_1 \ldots A_n \ldots \in \tilde{\mathcal{F}} \Longrightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$



(i) $\tilde{\Omega} = X^{-1}(\Omega) = \{\omega \in \tilde{\Omega} : X(\omega) \in \Omega\}$

$\mathcal{F}$ is σ-field on $\Omega \Rightarrow \Omega \in \mathcal{F} \Rightarrow X^{-1}(\Omega) \in \tilde{\mathcal{F}}$ by definition

(ii) $A \in \tilde{\mathcal{F}} \Rightarrow A = X^{-1}(B)$ where $B \in \mathcal{F}$ by definition

$$A^c = [X^{-1}(B)]^c$$
$$= X^{-1}(B^c)$$

But since $\mathcal{F}$ is a σ-field & $B \in \mathcal{F} \Rightarrow B^c \in \mathcal{F}$

$\therefore$ by def of $\tilde{\mathcal{F}} \Rightarrow X^{-1}(B^c) \in \tilde{\mathcal{F}}$

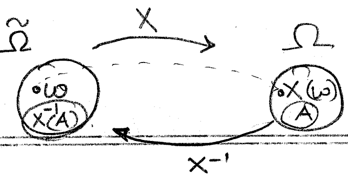
(iii) $A_1, A_2 \ldots \in \tilde{\mathcal{F}} \rightarrow$ $\left.\begin{array}{l} A_1 = X^{-1}(B_1) \\ A_2 = X^{-1}(B_2) \\ \vdots \end{array}\right\}$ for some $B_1, B_2 \ldots \in$

Now since $B_1, B_2, \ldots \in \mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} B_n \in \mathcal{F}$

$\therefore X^{-1}\left(\bigcup_{n=1}^{\infty} B_n\right) \in \tilde{\mathcal{F}}$ by def of $\tilde{\mathcal{F}}$

$$\Rightarrow \bigcup_{n=1}^{\infty}\left(X^{-1}(B_n)\right) \in \tilde{\mathcal{F}}$$

$$\Rightarrow \bigcup_{n=1}^{\infty}(A_n) \in \tilde{\mathcal{F}}$$



7] Ω and $\tilde{\Omega}$ are arbitrary sets

$X : \tilde{\Omega} \longrightarrow \Omega$ is any set function.

$\tilde{\mathcal{F}}$ is a σ-field on $\tilde{\Omega}$

w $\mathcal{F} = \{ A \subset \Omega \,,\ X^{-1}(A) \in \tilde{\mathcal{F}} \}$ is a σ-field on Ω

Check (i) $\Omega \in \mathcal{F}$

(ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

(iii) $A_1, A_2 \dots \in \mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$

(i) $\Omega \subseteq \Omega$ and $X^{-1}(\Omega) = \tilde{\Omega}$ & we know $\tilde{\Omega} \in \tilde{\mathcal{F}}$ {since $\tilde{\mathcal{F}}$ is σ-field

$\Rightarrow X^{-1}(\Omega) \in \tilde{\mathcal{F}}$

$\therefore$ $\Omega \in \mathcal{F}$ by def of $\mathcal{F}$

(ii) $A \in \mathcal{F} \Rightarrow A \subset \Omega$ and $X^{-1}(A) \in \tilde{\mathcal{F}}$

Also ~~consider~~ $A^c \subset \Omega$ {since $A \subset \Omega$}

also $X^{-1}(A^c) = (X^{-1}(A))^c$ which $\in \tilde{\mathcal{F}}$ {also since $\tilde{\mathcal{F}}$ is σ-field & $(X^{-1}(A)) \in \tilde{\mathcal{F}}$

$\therefore$ we have $A^c \subset \Omega$ & $X^{-1}(A^c) \in \tilde{\mathcal{F}} \Rightarrow A^c \in \mathcal{F}$

(iii) $A_1, A_2 \dots \in \mathcal{F}$ we must show $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$

ie to show $\bigcup_{n=1}^{\infty} A_n \subset \Omega$ and $X^{-1}\left(\bigcup_{n=1}^{\infty} A_n\right) \in \tilde{\mathcal{F}}$

Now $A_1, A_2 \dots \in \mathcal{F} \Rightarrow A_1 \subset \Omega, A_2 \subset \Omega, \dots A_n \subset \Omega \dots$

$\Rightarrow \left(\bigcup_{n=1}^{\infty} A_n\right) \subset \Omega$

Now $X^{-1}\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty}\left(X^{-1}(A_n)\right)$

Now all $A_1, A_2 \dots \in \mathcal{F} \Rightarrow X^{-1}(A_1), X^{-1}(A_2) \dots \in \tilde{\mathcal{F}}$

$\Rightarrow \bigcup_{n=1}^{\infty}\left(X^{-1}(A_n)\right) \in \tilde{\mathcal{F}}$ {since $\tilde{\mathcal{F}}$ is σ-field}

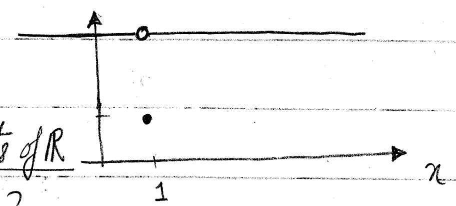
$\Rightarrow X^{-1}\left(\bigcup_{n=1}^{\infty} A_n\right) \in \tilde{\mathcal{F}}$

② Pg $f: \mathbb{A} \to \mathbb{B}$ $\quad f^{-1}(\mathbb{B}) = \mathbb{A}$ always But $f(\mathbb{A}) = \mathbb{B}$ is not al true

eg: $f: \mathbb{R} \to \mathbb{R}$
st $f(x) = 5 \Rightarrow f^{-1}(5) = \mathbb{R}$

Counter example

#2] If we have a continuous funct^n $f: \mathbb{R} \to \mathbb{R}$
can we say f is measurable ? Yes !
But every measurable funct^n is not continuous.
eg $\quad f(x) = \begin{cases} 1 & x = 1 \\ 5 & \text{OW} \end{cases}$ is measurable but not co



Now $\mathcal{B} \equiv \sigma(\mathcal{C}_1)$; $\mathcal{C}_1$ = class of all open sets of $\mathbb{R}$
$\equiv \sigma(\mathcal{C}_2)$ $\quad \mathcal{C}_2 = \{[a, \infty) : a \in \mathbb{R}\}$
$\equiv \sigma(\mathcal{C}_3)$ $\quad \mathcal{C}_3 = \{(a, \infty) : a \in \mathbb{R}\}$

To show f is measurable it is enough to show $f^{-1}(A) \in \mathcal{B}$ ∀

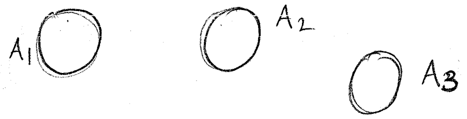
Now $f^{-1}(A) = f^{-1}[a, \infty)$ since $A \in \mathcal{C}_2$ it is of the type [
If $a > 5$ then $f^{-1}[a, \infty) = \phi$
If $1 < a \leq 5$ then $f^{-1}[a, \infty) = \{x : f(x) \in [a, \infty)\} = \mathbb{R} \setminus \{$
If $a \leq 1$ then $f^{-1}[a, \infty) = \{x : f(x) \in [a, \infty)\} = \mathbb{R}$

ϕ, $\mathbb{R} \setminus \{1\}$ & $\mathbb{R}$ are all $\mathcal{B}$ sets $\Rightarrow f^{-1}(A) \in \mathcal{B}$ ∀ $A \in$
$\Rightarrow f$ is measurable

$\{1\}$ is a closed set since $\{1\}^c$ is open and so measurable.

#2) $X(\omega) = a_1 I_{A_1}(\omega) + a_2 I_{A_2}(\omega) + \cdots + a_n I_{A_n}(\omega)$
where A_i's are disjoint and n is finite & $A_i \in \mathcal{A}$ (
{ The union $\cup A_i$'s does not have to be Ω
$f(x) = a_1 I_{A_1}(x) + \cdots + a_n I_{A_n}(x) + a_{n+1} I_{(\cup A_i)^c}(x)$ }

?. Now f can take only $(n+1)$ distinct values



f takes only finite # of values

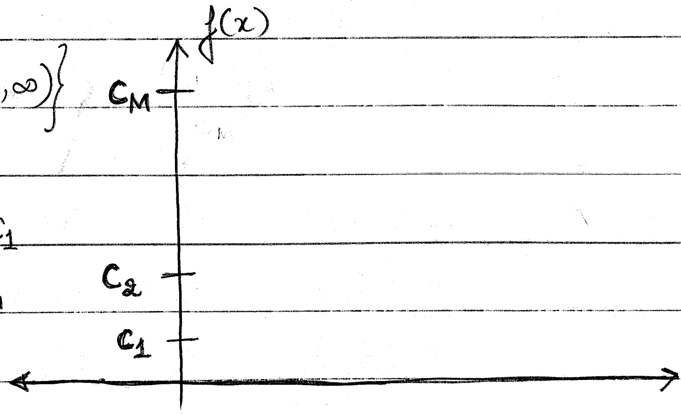
eg $f(x) = 3\, I_{[0,1]}(x) + 5\, I_{[3,7)}(x)$

$f(x) = \begin{cases} 3 & \text{if } x \in [0,1] \\ 5 & \text{if } x \in [3,7] \\ 0 & \text{ow} \end{cases}$

If $\bigcup_{i=1}^{n} A_i = \Omega$, f takes n values; ow f takes $n+1$ values

We need to show $f^{-1}(A) \in \mathcal{A} \quad \forall A \in \mathcal{C}_2 = \{[a,\infty) \,;\, a \in \mathbb{R}\}$

Now let f take values $c_1 < c_2 < \dots < c_M$ & assume $\bigcup_{i=1}^{n} A_i = \Omega$



If $a \le c_1 \Rightarrow f^{-1}([a,\infty)) = \{x : f(x) \in [a,\infty)\} = \mathbb{R}$

since all values of $f(x) \geq c_1$

if we take any $x_0 \in \mathbb{R} \Rightarrow f(x_0) \geq c_1$

If $c_1 < a \le c_2$ then $f^{-1}[a,\infty) = \mathbb{R} \setminus A_{n_0}$ for some $n_0 \in \{1,2,\dots n\}$

If $c_2 < a \le c_3$ then $f^{-1}[a,\infty) = \mathbb{R} \setminus (A_{n_0} \cup A_{n_1}) = (A_{n_0} \cup A_{n_1})^c \in \mathcal{A}$

$\vdots$

If $a > c_M$ then $f^{-1}[a,\infty) = \phi \quad \in \mathcal{A}$

each of the above $f^{-1}[a,\infty)$ are measurable

$\Rightarrow f^{-1}[a,\infty) \in \mathcal{A}$

NOTE $f(x) : \mathbb{R} \longrightarrow \mathbb{R}$

A's must be from some σ-algebra (need not be $\mathcal{B}$)

We want to show $f(x) = \sum_{i=1}^{\infty} x_i I_{A_i}(x)$ is measurable

A_i's are disjoint

$\bigcup A_i \equiv \Omega$

Claim 0: $f(x) = \lim_{n\to\infty} f_n(x)$ $\Rightarrow$ An elementary funct^n can be expressed as limits of seq of simple

where $f_n(x) = \sum_{i=1}^{n} x_i \, I_{A_i}(x)$

Claim 1 f_n's are measurable since f_n's are simple funct

Claim 2 limit of a measurable function is also measurable

$\therefore$ $f(x) = \sum_{i=1}^{\infty} x_i \, I_{A_i}(x)$ is measurable

#4] $\underline{\lim} \, A_n = \bigcup_{n=1}^{\infty} \bigcap_{m\geq n} A_m$; where A_i's are sets

$\overline{\lim} \, A_n = \bigcap_{n=1}^{\infty} \bigcup_{m\geq n}^{\infty} A_m$

Note If we have a seq of # which is monotonic then th have a limit (finite or infinite)

$\liminf x_n = \sup_n \left(\inf_{m\geq n} (x_m) \right)$

As $n\uparrow$ inf of x_m will $\uparrow$ eg $\{1, 2 \cdots \infty\}$ has inf 1

$\{2, 3, \cdots \infty\}$ has inf 2

$\{3, 4, \cdots \infty\}$ has inf 3

$\vdots$

And $\limsup x_n = \inf_n \left(\sup_{m\geq n} (x_m) \right)$

As $n\uparrow$ sup of x_m will $\downarrow$ & its lim is lower bdd

Result

If $\omega \in \underline{\lim} A_n \Rightarrow \omega \in \bigcup_{n=1}^{\infty} \bigcap_{m \geq n} A_m$

$\Rightarrow \exists\, n_0$ st $\omega \in \bigcap_{m \geq n_0} A_m$

$\Rightarrow \omega \in A_m \quad \forall\, m \geq n_0$

ie $\omega \in A_m \ \forall\, m$ except finitely many values

If $\omega \in \overline{\lim} A_n \Rightarrow \omega \in \bigcap_{n=1}^{\infty} \bigcup_{m \geq n} A_m$

$\Rightarrow \omega \in \bigcup_{m \geq n} A_m \quad \forall\, n$

given any $M \Rightarrow \exists$ some N_0 st $\omega \in A_{N_0}$

$\therefore$ $\omega \in$ infinitely many A_n's (without exception)

ie $\underline{\lim} A_n \subset \overline{\lim} A_n$ (all but finitely many is a special case of all)

Back to #4

$$A_n = \begin{cases} \left(\frac{1}{n}, \frac{2}{3} - \frac{1}{n}\right) & ; n = 5, 7, 9 \ldots \\ \left(\frac{1}{3} - \frac{1}{n}, 1 + \frac{1}{n}\right) & ; n = 2, 4, 6, 8 \ldots \end{cases}$$

Let

$A_1 = \mathbb{R}$ & $A_3 = \mathbb{R}$ {altering finite # of sets does not change the limits

$$\underline{\lim} A_n = \bigcup_{n=1}^{\infty} \bigcap_{m \geq n} A_m$$

$$= \bigcup_{n=1}^{\infty} \left[\left(\bigcap_{\substack{m \geq n \\ m: odd}} A_m \right) \cap \left(\bigcap_{\substack{m \geq n \\ m: even}} A_m \right) \right]$$

Pg 4

Now $\bigcap_{\substack{m \geq n \\ m: odd}} A_m = \bigcap_{\substack{m \geq n \\ n: odd}} \left(1/n, \ 2/3 - 1/n\right) = \left(1/n, 2/3 - 1/n\right)$ {wh n

since it is a ↑ seq sets the $\bigcap$ is the 1st

$\bigcap_{\substack{m \geq n \\ m: even}} = \bigcap_{\substack{m \geq n \\ m: even}} \left(1/3 - 1/n, \ 1 + 1/n\right) = \left[\frac{1}{3}, 1\right]$

$\therefore \bigcap_{m \geq n} A_m = \left(1/n, \ 2/3 - 1/n\right) \cap \left[1/3, 1\right]$ $\begin{cases} n \text{ is odd} \\ n \geqslant 5 \end{cases}$

$= \left[1/3, \ 2/3 - 1/n\right)$

$\therefore \bigcup_{n=1}^{\infty} \bigcap_{m \geq n} A_m = \bigcup_{n=1}^{\infty} \left[1/3, \ 2/3 - 1/n\right) = \left[1/3, 2/3\right)$.

$\overline{\lim} A_n = \bigcap_{n=1}^{\infty} \bigcup_{m \geq n}^{\infty} A_m$

$= \bigcap_{n=1}^{\infty} \left[\left(\bigcup_{\substack{m \geq n \\ m: even}}^{\infty} A_m \right) \cup \left(\bigcup_{\substack{m \geq n \\ m: odd}}^{\infty} A_m \right) \right]$

Now $\bigcup_{\substack{m \geq n \\ m: even}}^{\infty} A_m = \underbrace{\bigcup_{\substack{m \geq n \\ m: even}}^{\infty} \left(1/3 - 1/m, \ 1 + 1/m\right)}_{\text{decreasing seq of sets}} = \left(1/3 - 1/n, \ 1 + 1/n\right)$

$\bigcup_{\substack{m \geq n \\ m: odd}} A_m = \bigcup_{\substack{m \geq n \\ m: odd}}^{\infty} \left(1/m, \ 2/3 - 1/m\right) = \left(0, 2/3\right)$

$\therefore \overline{\lim} A_n = \bigcap_{n=1}^{\infty} \left[\left(0, 2/3\right) \cup \left(1/3 - 1/n, \ 1 + 1/n\right) \right]$ {n

$= \bigcap_{n=1}^{\infty} \left(0, 1 + 1/n\right) = \left(0, 1\right]$?

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu\left(\bigcup_{m=1}^{n} A_m\right) \quad \{\text{Prop } 1.2\}$$

5

i) $$\mu(\underline{\lim}\, A_n) = \mu\left(\bigcup_{n=1}^{\infty}\left(\bigcap_{m\geq n}^{\infty} A_m\right)\right)$$

$$\leq \sum_{n=1}^{\infty} \mu\left(\bigcap_{m\geq n}^{\infty} A_m\right) \quad \{\text{Prop } 1.2 \text{ pg } 8$$

$$\leq \sum_{n=1}^{\infty} \mu(A_n)$$

$\bigcap_{m\geq n}^{\infty} A_m$ is a $\uparrow$ seq

as you intersect fewer terms the intersection gets bigger

Let $B_n \equiv \bigcap_{m\geq n}^{\infty} A_m$ which is an $\uparrow$ seq

$$\underline{\lim}\, A_n = \bigcup_{n=1}^{\infty}(B_n) = \lim_{n\to\infty}(B_n)$$

$A_1 \cap A_2 \cap A_3 \cdots$

$A_2 \cap A_3 \cap A_4 \cdots$

$$\begin{cases} \mu(\underline{\lim}\, A_n) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) \\ \qquad = \lim_{n\to\infty} \mu(B_n) \quad \{\text{Prop } 1.2 \text{ since } B_n \uparrow \text{seq} \end{cases}$$

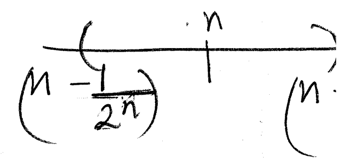
Now $\mu(B_n) = \mu\left(\bigcap_{m\geq n}^{\infty} A_m\right) \leq \mu(A_n)$ $\left\{\text{since } \bigcap_{m\geq n}^{\infty} A_m \subset A_n\right.$

$$\Rightarrow \underline{\lim}\, \mu(B_n) \leq \underline{\lim}\, \mu(A_n)$$

$$\Rightarrow \lim \mu(B_n) \leq \underline{\lim}\, \mu(A_n)$$

$\left\{\text{since Lim of } B_n \text{ exists}\right.$ $\lim B_n = \underline{\lim}\, B_n = \overline{\lim}\, B_n$

$$\Rightarrow \mu(\underline{\lim}\, A_n) \leq \underline{\lim}\, \mu(A_n)$$



#10] $\{x : |x-n| < 1/2^n$ for some $n \in \mathbb{N}\}$
$\equiv \bigcup_{n=1}^{\infty} \{x : |x-n| < 1/2^n\}$ } show this $\supseteq$ & $\subseteq$ arguement.

$\therefore \lambda\{x : |x-n| < 1/2^n\} = \lambda\left(\bigcup_{n=1}^{\infty}\{x : |x-n| < 1/2^n\}\right).$

$= \sum_{n=1}^{\infty} \lambda(\{x : |x-n| < 1/2^n)$ {dis se

$= \sum_{n=1}^{\infty} \frac{2}{2^n}$

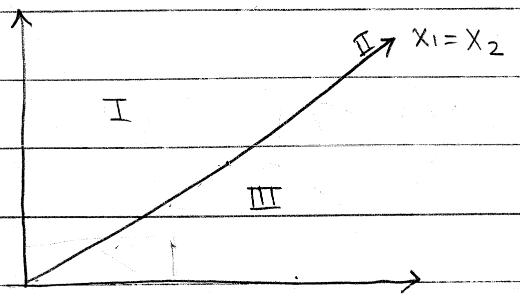
$= 2.$

#10(b)) We need to show
$\delta(\omega) \geq 0$
$\delta_\omega(\Omega) = 1.$
$\delta_\omega(\phi) = 0$
$\delta.(\Sigma A) = \Sigma \delta_\omega(A)$

Largest set of measure zero $\delta_{\omega_0}(A) = \begin{cases} 1 & \omega_0 \in A \\ 0 & OW \end{cases}$ fixed

$A = \Omega \setminus \{\omega_0\}$ is the largest set of measure zero
$B = \{\omega_0\}$ is the largest set of measure one.

$$Z_2 = \begin{cases} 1 & \text{if} \quad x_1 > x_2 \\ 0 & \text{if} \quad x_1 = x_2 \\ -1 & \text{if} \quad x_1 < x_2 \end{cases} \qquad \text{ie} \quad Z_2 = \text{sign}(x_1 - x_2)$$



$Z_2 : (X_1, X_2) \longrightarrow \mathbb{R}$

$$\mathcal{F}(Z_2) = \{ Z_2^{-1}(A) : A \in \mathcal{B} \}$$

For any $A \in \mathcal{B}$ either $1, 0, -1$ belongs to A

$$Z_2^{-1}(A) = \begin{cases} \phi & \text{if} \quad 0, 1, -1 \notin A \\ \text{II} \cup \text{III} & \text{if} \quad 0 \text{ or } 1 \in A \\ \text{I} \cup \text{II} & \text{if} \quad 0, -1 \in A \\ \text{I} \cup \text{III} & \text{if} \quad 1, -1 \in A \\ \text{I} \cup \text{III} \cup \text{II} & \text{if} \quad 1, 0, -1 \in A \end{cases}$$

$\therefore$ $\mathcal{F}(Z_2)$ = set of finite unions of I, II, III with ϕ included

$$Z_1(X_1, X_2) = \sqrt{X_1^2 + X_2^2}$$

$$\mathcal{F}(Z_1) = \{ Z^{-1}(A) : A \in \mathcal{B} \}$$
$= \sigma \{$generated by $Z^{-1}(\mathcal{C})\}$ where $\mathcal{B} = \sigma[\mathcal{C}]$

Pg 22 $= \sigma \{ Z^{-1}(-\infty, a] \}$ letting $\mathcal{C} = (-\infty, a]$

$= \sigma \{ (x_1, x_2) : Z_1(x_1, x_2) \in (-\infty, a] \}$

$= \sigma \{ (x_1, x_2) : (x_1^2 + x_2^2) \in (-\infty, a] \} = \sigma \{ (x_1, x_2) : (x_1^2 + x_2^2) \leq a \}$

$= \sigma \{$generated by all discs of radius $a\}$

$\mathcal{F}(Z_1) = \sigma\{$generated by all discs of radius a$\}$

$\mathcal{F}(Z_1, Z_2) = \sigma\{$ generated by I, II, III and discs of radius a $\}$