8.

Let $\Omega' \equiv \{1,2,3,4\}$ and $\Omega \equiv \{a,b,d\}$

$\mathcal{F}' \equiv \{\Omega', \{1,2,4\}, \{3\}, \emptyset\}$

Note that $\mathcal{F}'$ is a σ-field

Let $X(1) = a$

$X(2) = b$

$X(3) = a$

$X(4) = d$

It follows that

$X\{\mathcal{F}'\} \equiv \{X(A) : A \in \mathcal{F}'\} \equiv \{\Omega, \{a\}, \emptyset\}$

note that $\{a\} \in X\{\mathcal{F}'\}$

however, $\{a\}^c \equiv \{b,d\} \notin X\{\mathcal{F}'\}$

Therefore, $X\{\mathcal{F}'\}$ is not a σ-field.

9. $z_1 = \sqrt{(x_1)^2 + (x_2)^2}$

$\mathcal{F}(z_1) \equiv \sigma\{ z_1^{-1} A : A \in \mathcal{B}\}$ = { rotation wrt (0,0) of any set in $\mathcal{B}$ }

$\equiv \sigma\{(x_1,x_2) : \sqrt{(x_1)^2 + (x_2)^2} \in (-\infty,a] \}$

$\equiv \sigma\{(x_1,x_2) : \sqrt{(x_1)^2 + (x_2)^2} \in [0,a] \}$

$\equiv \sigma\{(x_1,x_2) : \sqrt{(x_1)^2 + (x_2)^2} \le a\}$

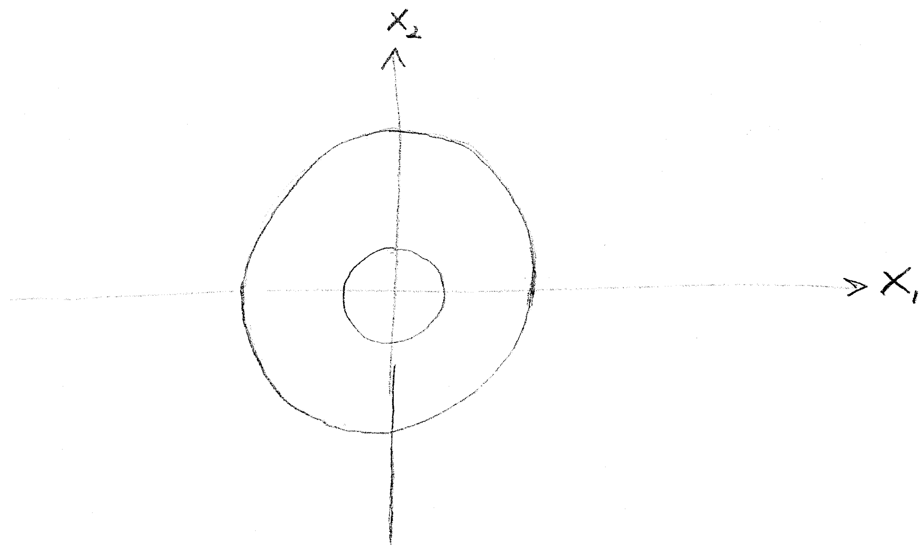
$\equiv \sigma\{(x_1,x_2) : (x_1)^2 + (x_2)^2 \le a^2\}$

$\mathcal{B} = \sigma\{(-\infty, a] : a \in \mathbb{R}\}$

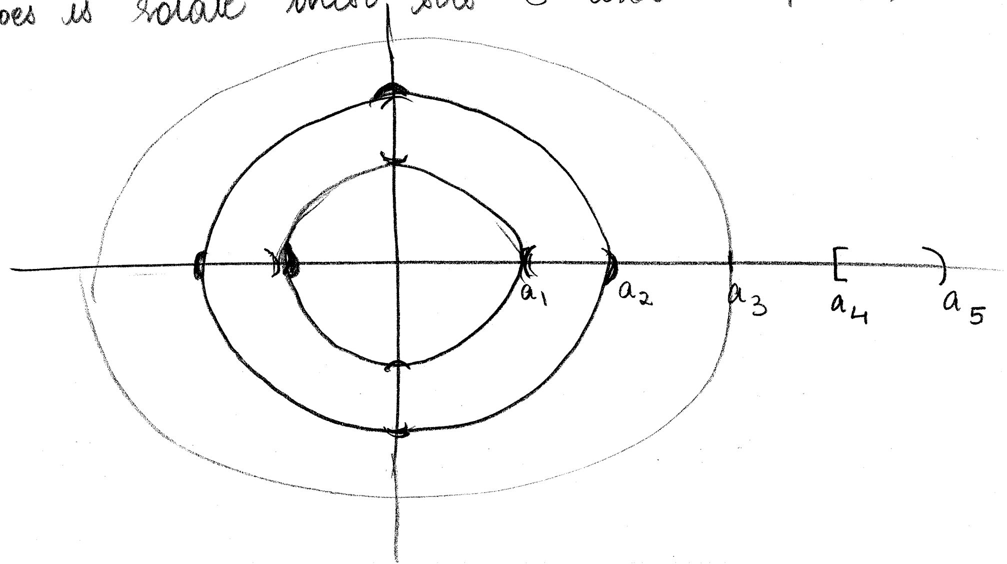
$\sigma(X) = X^{-1}\sigma(\mathcal{B})$

$\{X^{-1}B : B \in \mathcal{B}\}$ = pg 22 (13)

note that a is any real number implies that a^2 is any real number
Therefore, $\mathcal{F}(z_1)$ is a σ-field generate by a class of sets where each set can be represented as disc with an arbitrary radius a^2 $\forall\ 0 \le a^2 \le \infty$.



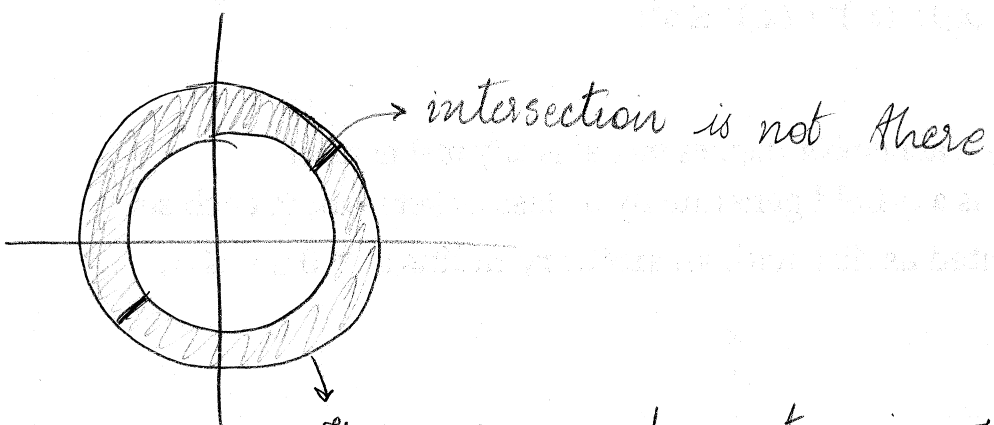
$\mathcal{B} := \{ (a_1,a_2) \cup \{a_3\} \cup [a_4,a_5) \}$ what $Z^{-1}(B)$, $B \in \mathcal{B}$ does is rotate these sets B around pt (0,0)



$\mathcal{F}(Z_1) = \{ Z_1^{-1}(B) : B \in \mathcal{B} \}$

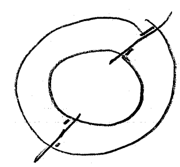
$\mathcal{F}(Z_2) = \{ Z_2^{-1}(C) : C \in \mathcal{B} \}$

$\mathcal{F}(Z_1, Z_2) = \sigma \{ Z_1^{-1}(B) \cup Z_2^{-1}(B) \}$



This is an element in $\mathcal{F}(Z_1, Z_2)$

but not in $\mathcal{F}(Z_1)$



Number 9 continued

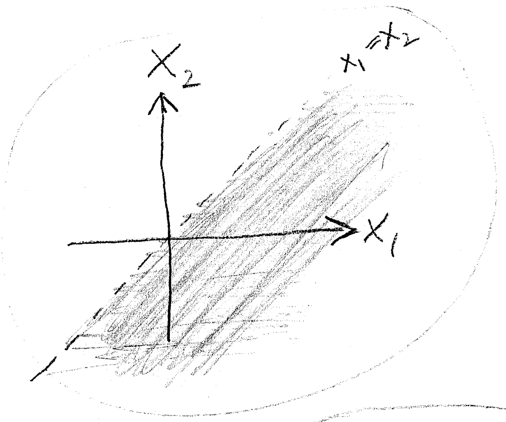
$$Z_2 = \text{sign}(x_1 - x_2) = \begin{cases} 1 & \text{if } x_1 > x_2 \text{ corresponds to region III} \\ 0 & \text{if } x_1 = x_2 \text{ corresponds to region II} \\ -1 & \text{if } x_1 < x_2 \text{ corresponds to region I} \end{cases}$$

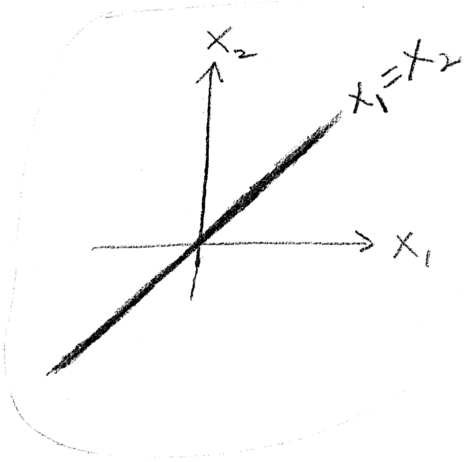
$$\mathcal{F}(z_2) \equiv \sigma\{ z_2^{-1} A : A \in \mathcal{B}\}$$

$$\equiv \sigma\{ z_2^{-1}\{1\}, z_2^{-1}\{0\}, z_2^{-1}\{-1\}, z_2^{-1}\{1,0\}, z_2^{-1}\{1,-1\}, z_2^{-1}\{0,-1\}, z_2^{1}\{1,0,1\}\}$$

$$\equiv \sigma\{ \text{III}, \text{II}, \text{I}, \text{III}\cup\text{II}, \text{III}\cup\text{I}, \text{II}\cup\text{I}, \text{III}\cup\text{II}\cup\text{I}\}$$

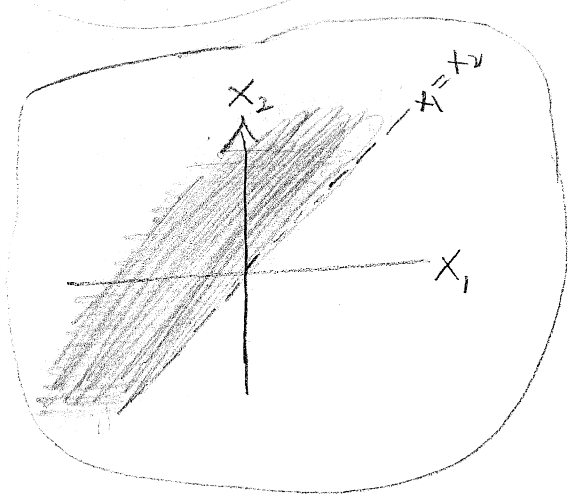
III ≡





≡ II

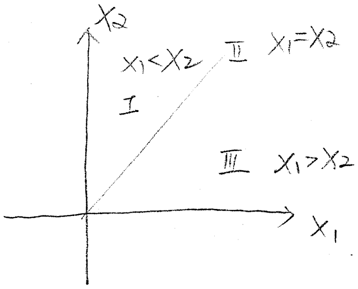
I ≡

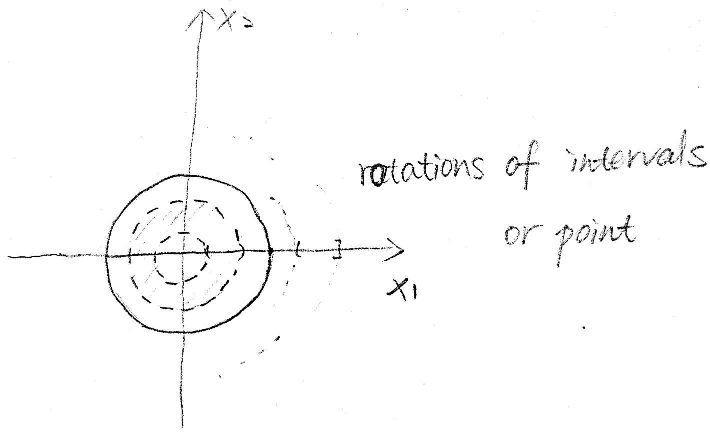


HW #2

7. $\mathcal{F}(Z_1, Z_2) = \sigma\{Z_1^{-1}B \cup Z_2^{-1}C,\ B, C \in \mathcal{B}\}$

$= \sigma[\{$the three parts in ①$\} \cup \{$all the rotations in ②$\}]$





10. a) $\{x : |x-n| < \frac{1}{2^n}$ for some $n \in \mathbb{N}\} = \bigcup_{n=1}^{\infty} \{n : |x-n| < \frac{1}{2^n}\}$

So $\lambda(\bigcup_{n=1}^{\infty} \{n : |x-n| < \frac{1}{2^n}\}) = \sum_{n=1}^{\infty} \lambda\{n : |x-n| < \frac{1}{2^n}\} = \sum_{n=1}^{\infty} \frac{2}{2^n} = \frac{1}{1-\frac{1}{2}} = 2$

b) ① $\delta_w(\phi) = 0$. $\delta_w(\Omega) = 1$.

② Let $A_1, A_2 \cdots A_n \cdots \in \Omega$. disjoint. we need to show that $\delta_w(\sum_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \delta_w(A_n)$

1) $w \in \bigcup_{n=1}^{\infty} A_n$. So $\exists n_0 \in N \ni w \in A_{n_0}$ $\quad w \notin \bigcup_{\substack{n=1 \\ n \neq n_0}}^{\infty} A_{n_0}$

So $\delta_w(\sum_{n=1}^{\infty} A_n) = \delta_w(A_{n_0}) = 1$

$\sum_{n=1}^{\infty} \delta_w(A_n) = \sum_{\substack{n=1 \\ n \neq n_0}}^{\infty} \delta_w(A_n) + \delta_w(A_{n_0}) = 0 + 1 = 1$

$\therefore \delta_w(\sum_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \delta_w(A_n)$

2) $w \notin \bigcup_{n=1}^{\infty} A_n$. $\quad \delta_w(\sum_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \delta_w(A_n) = 0$

So $\delta_w : A \subseteq \Omega \to [0, 1]$ is a probability measure

the largest set of measure zero is $\Omega \setminus \{w\}$

the largest set of measure 1 is Ω.