1] **Slutsky Theorem** : If $X_n \xrightarrow{d} X$, $Y_n \xrightarrow{P} a$, $Z_n \xrightarrow{P} b$ $\Rightarrow$ $X_n Y_n + Z_n \xrightarrow{d} aX + b$

Proof : Step 1 : If $U_n - V_n \xrightarrow{P} 0$, $U_n \xrightarrow{d} U$ $\Rightarrow$ $V_n \xrightarrow{d} U$

Step 2 : If $X_n \xrightarrow{d} X$, $Y_n \xrightarrow{P} a$ $\Rightarrow$ $X_n Y_n \xrightarrow{d} aX$

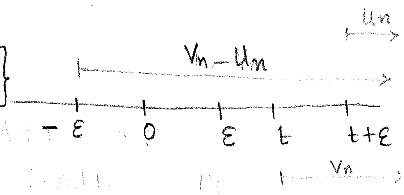
Step 3 : If $X_n \xrightarrow{d} X$, $Z_n \xrightarrow{P} b$ $\Rightarrow$ $X_n + Z_n \xrightarrow{d} X + b$

Step 1 : To show $V_n \xrightarrow{d} U$ we have to show $F_{V_n}(t) = P(V_n \le t) \to F_U(t)$

We will show $F_U(t) \le \underline{\lim}\, F_{V_n}(t) \le \overline{\lim}\, F_{V_n}(t) \le F_U(t)$

Now $F_{V_n}(t) = P(V_n \le t) = P(V_n - U_n + U_n \le t + \varepsilon - \varepsilon)$

$$\left.\begin{array}{l} V_n - U_n \ge -\varepsilon \\ U_n \ge t+\varepsilon \end{array}\right\} \Rightarrow V_n \ge t \left[\text{ie } \{V_n - U_n \ge -\varepsilon\} \cap \{U_n \ge t+\varepsilon\} \subseteq \{V_n \ge t\} \right.$$



$\therefore$ $P(V_n \ge t) \ge P(V_n - U_n \ge -\varepsilon \text{ and } U_n \ge t+\varepsilon)$

$\therefore$ $P(V_n \le t) \le P(V_n - U_n \le -\varepsilon \ \cup \ U_n \le t+\varepsilon)$ $\{ P(A^c \text{ OR } B^c) \ge P(C^c)$

$\le P(V_n - U_n \le -\varepsilon) + P(U_n \le t+\varepsilon)$

ie $P(V_n \le t) \le P(\varepsilon \le U_n - V_n) + P(U_n \le t+\varepsilon)$

$\le P(|U_n - V_n| \ge \varepsilon) + P(U_n \le t+\varepsilon)$ $\left[\because \{|U_n - V_n| \ge \varepsilon\} \supset \{U_n - V_n \ge \varepsilon\}\right.$

$F_{V_n}(t) \le P(|U_n - V_n| \ge \varepsilon) + F_{U_n}(t+\varepsilon)$

$\overline{\lim}\, F_{V_n}(t) \le \overline{\lim}\, P(|U_n - V_n| \ge \varepsilon) + \overline{\lim}\, F_{U_n}(t+\varepsilon)$ ——— (*)

Now since $U_n - V_n \xrightarrow{P} 0$ $\Rightarrow$ $\underline{\lim}\, P(|U_n - V_n| \ge \varepsilon) = \overline{\lim}\, P(|U_n - V_n| \ge \varepsilon)$

$= \lim_{n\to\infty} P(|U_n - V_n| \ge \varepsilon) \le \varepsilon$

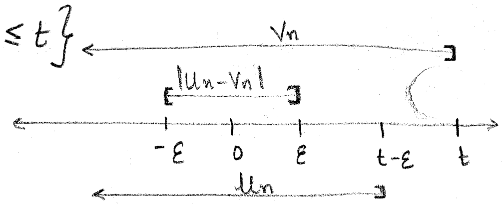
also $U_n \xrightarrow{d} U$ $\Rightarrow$ $\overline{\lim}\, F_{U_n}(t) = \underline{\lim}\, F_{U_n}(t) = \lim_{n\to\infty} F_{U_n}(t) = F_U(t)$

for all continuity pts of $F_U(\cdot)$

$\therefore$ (*) $\Rightarrow$ $\overline{\lim}\, F_{V_n}(t) \le \varepsilon + \lim_{n\to\infty} F_{U_n}(t+\varepsilon)$ ——— ①

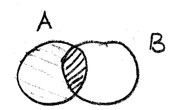
Now $F_{V_n}(t) = P(V_n \le t) = P(V_n - U_n + U_n \le t+\varepsilon-\varepsilon)$

$\{|V_n-U_n| \le \varepsilon\}$ and $\{U_n \le t-\varepsilon\} \subseteq \{V_n \le t\}$



∴ $P(V_n \le t) \ge P(U_n \le t-\varepsilon$ and $|V_n-U_n| \le \varepsilon)$

$$P(V_n \le t) \ge P(\underbrace{U_n \le t-\varepsilon}_{A} \cap \underbrace{|V_n-U_n| \le \varepsilon}_{B})$$



Now $P(A\cap B) = P(A) - P(A\cap B^c) \le P(A) - P(B^c)$

∴ $P(V_n \le t) \ge P(U_n \le t-\varepsilon) - P(|U_n-V_n| > \varepsilon)$

∴ $F_{V_n}(t) \ge F_{U_n}(t-\varepsilon) - P(|U_n-V_n| > \varepsilon)$

∴ $\underline{\lim}\, F_{V_n}(t) \ge \underline{\lim}\, F_{U_n}(t-\varepsilon) - \underline{\lim}\, P(|U_n-V_n| > \varepsilon)$

$$\underline{\lim}\, F_{V_n}(t) \ge \lim_{n\to\infty} F_{U_n}(t-\varepsilon) - \varepsilon \qquad \text{—— ②}$$

∴ $\lim\limits_{n\to\infty} F_{U_n}(t-\varepsilon) - \varepsilon \le \underline{\lim}\, F_{V_n}(t) \le \overline{\lim}\, F_{V_n}(t) \le \lim\limits_{n\to\infty} F_{U_n}(t+\varepsilon) + \varepsilon$

Let $\varepsilon \to 0$ and if t is a continuity pt of $F_U(\cdot)$ and also $t+\varepsilon$, $t-\varepsilon$ are continuity pts of $F_U(\cdot)$

∴ $F_U(t) \le \underline{\lim}\, F_{V_n}(t) \le \overline{\lim}\, F_{V_n}(t) \le F_U(t)$

∴ $\lim\limits_{n\to\infty} F_{V_n}(t) = F_U(t)$ where t is a continuity pt of $F_U(\cdot)$

ie $V_n \xrightarrow{d} U$

<u>Step 2</u> If $X_n \xrightarrow{d} X$, $Y_n \xrightarrow{P} a \Rightarrow X_nY_n \xrightarrow{d} aX$

We will first show $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{P} 0 \Rightarrow X_nY_n \xrightarrow{P} 0$

Now $P(|X_nY_n - 0| > \varepsilon) = P(|X_nY_n| > \varepsilon) = P(|X_nY_n| > \varepsilon, |Y_n| \le \varepsilon/k) + P(|X_nY_n| > \varepsilon, |Y_n| > \varepsilon/k)$

$$\le P(|X_n| > k) + P(|Y_n| > \varepsilon/k)$$

{ since $P(|X_nY_n| > \varepsilon, |Y_n| \le \varepsilon/k) \le P(|X_n| > k)$ and $P(A\cap B) \le P(B)$

$$\lim_{n\to\infty} P(|X_n Y_n| > \varepsilon) \leq \lim_{n\to\infty} P(|X_n| > k) + \lim_{n\to\infty} P(|Y_n| > \varepsilon/k) \quad \text{for some } k$$

$$\lim_{n\to\infty} P(|X_n Y_n| > \varepsilon) \leq \lim_{n\to\infty} P(|X_n| > k) \qquad \left\{ \because Y_n \xrightarrow{P} 0 \Rightarrow \lim_{n\to\infty} P(|Y_n| > \varepsilon/k) = 0. \right.$$

$$\lim_{n\to\infty} P(|X_n Y_n| > \varepsilon) \leq \lim_{n\to\infty} \left[1 - F_{X_n}(k)\right]$$

$$\leq 1 - \lim_{n\to\infty} F_{X_n}(k)$$

$$\leq 1 - F_X(k)$$

$$\leq 0$$

$$\Rightarrow \lim_{n\to\infty} P(|X_n Y_n| > \varepsilon) = 0 \Rightarrow X_n Y_n \xrightarrow{P} 0$$

Now If $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{P} a \iff X_n \xrightarrow{d} X$ and $(Y_n - a) \xrightarrow{P} 0$

$$\Rightarrow X_n (Y_n - a) \xrightarrow{P} 0$$

$\therefore$ we have $(X_n Y_n - X_n a) \xrightarrow{P} 0$ and $a X_n \xrightarrow{d} a X$

$$\Rightarrow X_n Y_n \xrightarrow{d} a X \qquad (\because \text{ by step 1})$$

Step 3 $\quad X_n \xrightarrow{d} X, \; Z_n \xrightarrow{P} b \Rightarrow X_n + Z_n \xrightarrow{d} X + b$

(i) $X_n \xrightarrow{d} X$ given ie $\lim_{n\to\infty} F_{X_n}(t) = F_X(t)$; at all continuity pts of $F_X(\circ)$

Now $\lim_{n\to\infty} P(X_n + b \leq t) = \lim_{n\to\infty} P(X_n \leq t - b) = P(X \leq t - b)$

$$= P(X + b \leq t)$$

$$\therefore X_n + b \xrightarrow{d} X + b.$$

Now
$$\left.\begin{array}{l} X_n + b \xrightarrow{d} X + b \\ (X_n + Z_n) - (X_n + b) = (Z_n - b) \xrightarrow{P} 0 \end{array}\right\} \Rightarrow X_n + Z_n \xrightarrow{d} X + b \quad [\because \text{from step 1}$$

$\therefore$ If $X_n \xrightarrow{d} X$, $Y_n \xrightarrow{P} a$ $\Rightarrow$ $X_n Y_n \xrightarrow{d} aX$

& then $Z_n \xrightarrow{P} b$ and $X_n Y_n \xrightarrow{d} aX$ $\Rightarrow$ $X_n Y_n + Z_n \xrightarrow{d} aX+b$

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2] Show $X_n \xrightarrow{\mu} X \iff X_m - X_n \xrightarrow{\mu} 0$. [We will only show it for $\mu(\Omega)<\infty$ case]

Proof $\Rightarrow$ given $X_n \xrightarrow{\mu} X \Rightarrow \mu(|X_n - X| > \varepsilon) \longrightarrow 0 \quad \forall \varepsilon > 0$

Now $|X_m - X_n| = |X_m - X + X - X_n|$

$$\leq |X_m - X| + |X_n - X|$$

$$\{|X_m - X| \leq \varepsilon/2\} \cap \{|X_n - X| \leq \varepsilon/2\} \subseteq \{|X_m - X_n| \leq \varepsilon\}$$

$$\{|X_m - X| > \varepsilon/2\} \cup \{|X_n - X| > \varepsilon/2\} \supseteq \{|X_m - X_n| > \varepsilon\}$$

$$\therefore \mu(|X_m - X_n| > \varepsilon) \leq \mu[(|X_m - X| > \varepsilon/2) \cup (|X_n - X| > \varepsilon/2)]$$

$$\leq \mu(|X_m - X| > \varepsilon/2) + \mu(|X_n - X| > \varepsilon/2)$$

$$0 \leq \lim_{n\to\infty} \mu(|X_m - X_n| > \varepsilon) \leq 0 + 0$$

$$\therefore (X_m - X_n) \xrightarrow{\mu} 0$$

$\Leftarrow$ given $X_m - X_n \xrightarrow{\mu} 0$ ie $\mu(|X_m - X_n| > \varepsilon) \leq \varepsilon \quad \forall m, n \geq n_\varepsilon$

(i) We can select a subseq $\{n_k\}\uparrow$ st $\mu(|X_m - X_{n_k}| > 1/2^k) \leq 1/2^k \quad \forall m \geq n_k$

[ie for $k=1$ $\mu(|X_m - X_{n_1}| > 1/2^k) \leq 1/2^k$ as long as $m \geq n_1 \geq n_{\varepsilon_1}$ ie set $n_\varepsilon = n_1$

for $k=2$ $\mu(|X_m - X_{n_2}| > 1/2^k) \leq 1/2^k$ as long as $m \geq n_2 \geq n_1 \geq n_{\varepsilon_2}$ ie st $n_2 = \max(n_1, n_{\varepsilon_2}$

(ii) Define $A_k \equiv \{|X_{n_k} - X_{n_{k+1}}| > 1/2^k\} \quad k \geq m$

$B_m \equiv \bigcup_{k=m}^{\infty} A_k$ [$B_m \downarrow$ seq

$C \equiv \bigcup_{m=1}^{\infty} B_m^c$

$$\mu(C^c) = \mu\left(\bigcap_{m=1}^{\infty} B_m\right) = \mu\left(\bigcap_{m=1}^{\infty} \bigcup_{k\geq m}^{\infty} A_k\right)$$

$$= \lim_{m\to\infty} \mu\left(\bigcup_{k=m}^{\infty} A_k\right) \qquad \{\because B_m \downarrow \text{ seq } \& \ \mu(\Omega) < \infty$$

$$\leq \lim_{m\to\infty} \sum_{k=m}^{\infty} \mu(A_k)$$

$$\mu(C^c) \leq \lim_{m \to \infty} \sum_{k=m}^{\infty} \frac{1}{2^k}$$

$$\mu(C^c) \leq \lim_{m \to \infty} \frac{1}{2^{m-1}} = 0$$

(iii) Now we will show $\{X_{n_k}\}$ is cauchy on C

$$\omega \in C \Rightarrow \omega \in \bigcup_{m=1}^{\infty} B_m^c$$

$$\Rightarrow \omega \in B_{m_0}^c \quad \text{for some } m_0$$

$$\Rightarrow \omega \in \bigcap_{k \geq m_0}^{\infty} A_k^c$$

$$\Rightarrow \omega \in A_k^c \quad \forall \; k \geq m_0$$

$$\Rightarrow \omega \in \left\{ \left| X_{n_k} - X_{n_{k+1}} \right| > \frac{1}{2^k} \right\}^c \quad \forall \; k \geq m_0$$

$$\Rightarrow \left| X_{n_k}(\omega) - X_{n_{k+1}}(\omega) \right| \leq \frac{1}{2^k} \quad \forall \; k \geq m_0$$

$\therefore \; \forall \; n_i, n_j > m_0$, $\left| X_{n_i} - X_{n_j} \right| = \left| X_{n_i} - X_{n_{i+1}} + X_{n_{i+1}} - X_{n_{i+2}} + X_{n_{i+2}} \cdots - X_{n_j} \right|$

$$\leq \left| X_{n_i} - X_{n_{i+1}} \right| + \left| X_{n_{i+1}} - X_{n_{i+2}} \right| + \cdots \left| X_{n_{j-1}} - X_{n_j} \right|$$

$$\leq \sum_{k=i}^{\infty} \left| X_{n_k} - X_{n_{k+1}} \right|$$

$$\leq \sum_{k=i}^{\infty} \frac{1}{2^k} = \frac{1}{2^{i-1}}$$

$$\Rightarrow \left| X_{n_i} - X_{n_j} \right| < \varepsilon \quad \forall \; n_i, n_j > m_0$$

$\therefore$ $\{X_{n_k}\}$ is cauchy on C and $\mu(C^c) = 0$

$\therefore$ $\{X_{n_k}\}$ is cauchy a.e

$\Rightarrow \lim_{n \to \infty} X_{n_k}$ exists a.e [every cauchy seq converges]

Let $\lim_{n \to \infty} X_{n_k} = X \quad \forall \; \omega \in C$

$= 0 \quad$ ow

(iv) Now $|X_n - X| \leq |X_n - X_{n_k}| + |X_{n_k} - X|$

$$\therefore \mu(|X_n - X| > \varepsilon) \leq \underbrace{\mu(|X_n - X_{n_k}| > \varepsilon)}_{\downarrow 0 \ (\because \text{cauchy})} + \underbrace{\mu(|X_{n_k} - X| > \varepsilon)}_{\downarrow 0 \ (\because X_{n_k} \xrightarrow{a.e} X \ \& \ \mu(\Omega) < \infty)}$$

$$\therefore 0 \leq \lim_{n\to\infty} \mu(|X_n - X| > \varepsilon) = 0$$

$$\therefore \quad X_n \xrightarrow{\mu} X$$

$\Leftarrow$ Plan
(i) Extract $\{X_{n_k}\}$ from $\{X_n\}$ st $\mu\{|X_{n_k} - X_m| > 1/2^k\} \leq 1/2^k$
(ii) Define a set C st $\mu(C^c) = 0$
(iii) Show for $w \in C$, $\{X_{n_k}\}$ is Cauchy & hence has a limit
(iv) Show $X_n \xrightarrow{\mu} X$

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Show $X_n \xrightarrow{d} a \iff X_n \xrightarrow{P} a$

Proof "$\Leftarrow$" given $X_n \xrightarrow{P} a \Rightarrow X_n \xrightarrow{d} a$ (by proposition)

"$\Rightarrow$" Given $X_n \xrightarrow{d} a$ to show $X_n \xrightarrow{P} a$ ie $P(|X_n - a| > \varepsilon) \longrightarrow 0$

Consider $P(|X_n - a| > \varepsilon) = P[(X_n - a > \varepsilon) \cup (X_n - a < -\varepsilon)]$

$= P[(X_n - a) > \varepsilon] + P[(X_n - a) < -\varepsilon]$ $\{\because$ disjoint union

$= P(X_n > a + \varepsilon) + P(X_n < a - \varepsilon)$

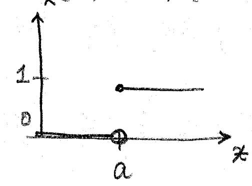
$= [1 - P(X_n \leq a + \varepsilon)] + P(X_n < a - \varepsilon)$

$\leq [1 - P(X_n \leq a + \varepsilon)] + P(X_n \leq a - \varepsilon)$ $\{\because \{X < c\} \subseteq \{X \leq c\}$

$P(|X_n - a| > \varepsilon) \leq 1 - F_{X_n}(a + \varepsilon) + F_{X_n}(a - \varepsilon)$ —— (*)

Now for $X = a$ $\quad F_X(x) = \begin{cases} 1 & \text{if } x \geq a \\ 0 & \text{oW} \end{cases}$

$F_X(x) = P(X \leq x) = P(a \leq x)$



Now $X_n \xrightarrow{d} X = a \iff F_{X_n}(x) \longrightarrow F_X(x)$ $\forall\, x$ in the continuity pts of $F_X(\cdot)$

Both $(a+\varepsilon)$ and $(a-\varepsilon)$ are continuity pts of $F_X(\cdot)$ $\Rightarrow$ $F_{X_n}(a+\varepsilon) \longrightarrow F_X(a+\varepsilon)$

& $F_{X_n}(a-\varepsilon) \longrightarrow F_X(a-\varepsilon)$

$(*) \Rightarrow$

$$\therefore \quad \underline{\lim}\ P(|X_n - a| > \varepsilon) \leq 1 - \lim F_{X_n}(a+\varepsilon) + \lim F_{X_n}(a-\varepsilon)$$

$$\therefore \quad 0 \leq \underline{\lim}\ P(|X_n - a| > \varepsilon) \leq 1 - F_X(a+\varepsilon) + F_X(a-\varepsilon)$$

$$0 \leq \underline{\lim}\ P(|X_n - a| > \varepsilon) \leq 1 - 1 + 0 = 0 \qquad \text{—— Ⓐ}$$

$|||^{rly}$ $(*) \Rightarrow$

$$\therefore \quad \overline{\lim}\ P(|X_n - a| > \varepsilon) \leq 1 - F_X(a+\varepsilon) + F_X(a-\varepsilon)$$

$$\therefore \quad 0 \leq \overline{\lim}\ P(|X_n - a| > \varepsilon) \leq 0 \qquad \text{—— Ⓑ}$$

Ⓐ & Ⓑ $\Rightarrow$ $\quad \underline{\lim}\ P(|X_n - a| > \varepsilon) = 0 = \overline{\lim}\ P(|X_n - a| > \varepsilon)$

$\Rightarrow \quad \lim\limits_{n \to \infty} P(|X_n - a| > \varepsilon) = 0$ and limit exists.

$\Rightarrow \quad X_n \xrightarrow{P} a$