S

Theorem.

$$\left.\begin{array}{l} X_n \xrightarrow{d} X \\ Y_n \xrightarrow{P} a \\ Z_n \xrightarrow{P} b \end{array}\right\} \Rightarrow Y_n \cdot X_n + Z_n \xrightarrow{d} aX+b$$

Step I. To prove $\boxed{\text{If } U_n - V_n \xrightarrow{P} 0 \text{ and } U_n \xrightarrow{d} U \text{ then } V_n \xrightarrow{d} U}$

① $F_{V_n}(t) = P(V_n \le t) = P(V_n - U_n + U_n \le t+\varepsilon-\varepsilon)$ (*)

since $\left.\begin{array}{l} V_n - U_n \geq -\varepsilon \\ U_n \geq t+\varepsilon \end{array}\right\} \Rightarrow V_n \geq t$

$\therefore$ (*) $\le P(V_n - U_n \le -\varepsilon$ and $U_n \le t+\varepsilon)$

$= P(U_n - V_n \geq \varepsilon$ and $U_n \le t+\varepsilon) \le P(|U_n - V_n| \geq \varepsilon) + P(U_n \le t+\varepsilon)$

$\overline{\lim_{n\to\infty}} F_{V_n}(t) \le \overline{\lim_{n\to\infty}} P(U_n \le t+\varepsilon) + \varepsilon = \overline{\lim} F_{U_n}(t+\varepsilon) + \varepsilon$

② $F_{V_n}(t) = P(V_n \le t) = P(V_n - U_n + U_n \le t+\varepsilon-\varepsilon)$

$\geq P(V_n - U_n \le \varepsilon$ and $U_n \le t-\varepsilon)$

$\geq P(|U_n - V_n| \le \varepsilon$ and $U_n \le t-\varepsilon)$

since $P(A\cap B) = P(A) - P(A\cap B^c) \geq P(A) - P(B^c)$

$\therefore \geq P(U_n \le t-\varepsilon) - P(|U_n - V_n| \le \varepsilon)$

$\underline{\lim} F_{V_n}(t) \geq \underline{\lim_{n\to\infty}} P(U_n \le t-\varepsilon) - \varepsilon = \underline{\lim_{n\to\infty}} F_{U_n}(t-\varepsilon) - \varepsilon$

$\underline{\lim} F_{U_n}(t-\varepsilon) - \varepsilon \le \underline{\lim} F_{V_n}(t) \le \overline{\lim} F_{V_n}(t) \le \overline{\lim} F_{U_n}(t+\varepsilon) + \varepsilon$

If t is the contimuity point of F_U by letting $\varepsilon \to 0$

the $t-\varepsilon$, and $t+\varepsilon$ are continuity points.

$\therefore F_U(t) \le \underline{\lim} F_{V_n}(t) \le \overline{\lim} F_{V_n}(t) \le F_U(t)$

$\therefore \lim F_{V_n}(t) = F_U(t)$

$\therefore V_n \xrightarrow{d} U$

Step II. To prove $X_n \xrightarrow{d} X,\ Y_n \xrightarrow{P} a \Rightarrow Y_n \cdot X_n \xrightarrow{d} aX$

ask

1. To prove $X_n \xrightarrow{d} X,\ Y_n \xrightarrow{P} 0 \Rightarrow Y_n \cdot X_n \xrightarrow{P} 0$

$P(|X_n \cdot Y_n| > \varepsilon) = P(|X_n \cdot Y_n| > \varepsilon,\ |Y_n| \le \frac{\varepsilon}{k}) + P(|X_n Y_n| > \varepsilon, |Y_n| > \frac{\varepsilon}{k})$

$\le P(|X_n| > k) + P(|Y_n| > \frac{\varepsilon}{k})$

$\Rightarrow \overline{\lim_{n\to\infty}} P(|X_n Y_n| > \varepsilon) \le P(|X| > k)$ for any fixed k

k is arbitrary, Hence $P(|X_n \cdot Y_n| > \varepsilon) \to 0$, $Y_n \cdot X_n \xrightarrow{P} 0$

Ask shuva

2. $X_n \cdot Y_n - aX_n = X_n(Y_n - a)$

$$\left.\begin{array}{l} \boxed{X_n \xrightarrow{d} X} \\ Y_n - a \xrightarrow{P} 0 \end{array}\right\} \Rightarrow X_n(Y_n - a) \xrightarrow{P} 0 \Rightarrow X_n \cdot Y_n \xrightarrow{d} aX$$

If $X_n \xrightarrow{d} X \Rightarrow aX_n \xrightarrow{d} aX$ (do we have to prove this)

$\boxed{\text{Step III}}$ To show $X_n \xrightarrow{d} X$, $Z_n \xrightarrow{P} b \Rightarrow X_n + Z_n \xrightarrow{d} X + b$ Ask Shuva

1. To show $X_n \xrightarrow{d} X \Rightarrow X_n + b \xrightarrow{d} X + b$

2. $$\left.\begin{array}{l} (X_n + Z_n) - (X_n + b) = Z_n - b \xrightarrow{P} 0 \\ X_n + b \xrightarrow{d} X + b \end{array}\right\} \Rightarrow \therefore X_n + Z_n \xrightarrow{d} X + b$$

$$\therefore \left.\begin{array}{l} X_n \xrightarrow{d} X \\ Y_n \xrightarrow{P} a \end{array}\right\} \Rightarrow \left.\begin{array}{l} Y_n \cdot X_n \xrightarrow{d} aX \\ Z_n \xrightarrow{P} b \end{array}\right\} \Rightarrow Y_n \cdot X_n + Z_n \xrightarrow{d} aX + b$$

$\underbrace{X+b}_{U_n} \xrightarrow{d} (X+b)$

$X_n \xrightarrow{d} X$

$$\Rightarrow \underbrace{X_n + b}_{V_n} \xrightarrow{d} \underbrace{X+b}_{U}$$

$(X_n + b) - b \xrightarrow{d} X$

$(X_n - b)$

<u>Slutsky's Lemma</u>: If $X_n \xrightarrow{d} X$, $Y_n \xrightarrow{P} a$, $Z_n \xrightarrow{P} b$ then

$$X_n Y_n + Z_n \xrightarrow{d} aX + b$$

<u>Proof</u>: <u>Step 1</u> Show if $\left.\begin{array}{l} U_n - V_n \xrightarrow{P} 0 \\ U_n \xrightarrow{d} U \end{array}\right\} \Rightarrow V_n \xrightarrow{d} U$

<u>Step 2</u> If $X_n \xrightarrow{d} X$, $Y_n \xrightarrow{P} a \Rightarrow X_n \cdot Y_n \xrightarrow{d} aX$

<u>Step 3</u> If $X_n \xrightarrow{d} X$, $Z_n \xrightarrow{P} b \Rightarrow X_n + Z_n \xrightarrow{d} X + b$

<u>Step 1</u> (prop 4.1, pg 33)

$$F_{V_n}(t) = P(V_n \leq t) = P(V_n - U_n + U_n \leq t + \varepsilon - \varepsilon) \quad \text{—(1)}$$

Now $\left.\begin{array}{l} V_n - U_n \geq -\varepsilon \\ U_n \geq t + \varepsilon \end{array}\right\} \Rightarrow V_n \geq t$

Since $P(B \cap C) \leq P(A^c) \Rightarrow P(B^c \cup C^c) \geq P(A)$

$$(1) \leq P(V_n - U_n \leq -\varepsilon \text{ OR } U_n \leq t + \varepsilon)$$

$$F_{V_n}(t) \leq P(V_n - U_n \leq -\varepsilon) + P(U_n \leq t + \varepsilon)$$

$\left\{ \left.\begin{array}{l} \text{let } B = [V_n - U_n \geq -\varepsilon] \\ \& \ C = [U_n \geq t + \varepsilon] \end{array}\right\} \Rightarrow \begin{array}{l} B \cap C \Rightarrow [V_n \geq t] \equiv A \\ \therefore P(B \cap C) \leq P(V_n \geq t) \end{array} \right\}$

$$\therefore F_{V_n}(t) \leq P(\varepsilon \leq U_n - V_n) + P(U_n \leq t + \varepsilon)$$

$$\leq P(|U_n - V_n| \geq \varepsilon) + P(U_n \leq t + \varepsilon)$$

Now $\overline{\lim_{n\to\infty}} F_{V_n}(t) \leq \lim_{n\to\infty} P(U_n \leq t + \varepsilon) + \varepsilon$ $\quad \{ \because U_n - V_n \xrightarrow{P} 0$

$$\overline{\lim_{n\to\infty}} F_{V_n}(t) \leq \lim_{n\to\infty} F_{U_n}(t + \varepsilon) + \varepsilon$$

$$F_{V_n}(t) = P(V_n \le t)$$
$$= P(V_n - U_n + U_n \le t + \varepsilon - \varepsilon)$$
$$\ge P(V_n - U_n \le \varepsilon \text{ and } U_n \le t - \varepsilon)$$
$$\ge P(|U_n - V_n| \le \varepsilon \text{ and } U_n \le t - \varepsilon$$
$$= P(A \cap B) = P(A) - P(A \cap B^c)$$
$$\ge P(A) - P(B^c)$$
$$\ge P(U_n \le t - \varepsilon) - P(|U_n - V_n| \le \varepsilon)$$

$$\underline{\lim} F_{V_n}(t) \ge \lim_{n\to\infty} P(U_n \le t - \varepsilon) - \varepsilon$$
$$= \lim_{n\to\infty} F_{U_n}(t - \varepsilon) - \varepsilon$$

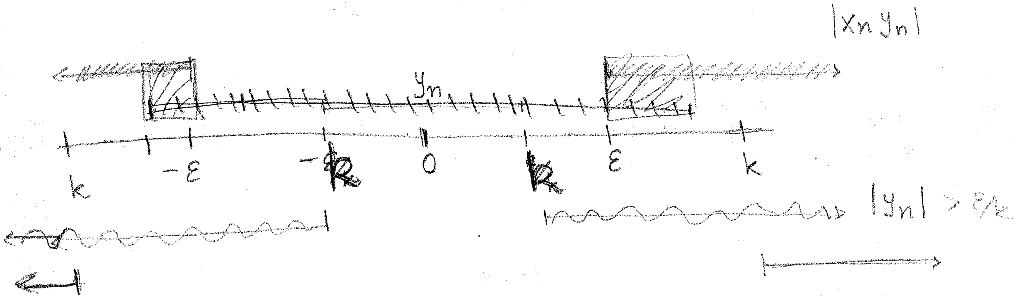
$$\lim_{n\to\infty} F_{U_n}(t-\varepsilon) - \varepsilon \le \underline{\lim} F_{V_n}(t) \le \overline{\lim} F_{V_n}(t) \le \lim F_{U_n}(t+\varepsilon) + \varepsilon$$
$$F_U(t-\varepsilon) - \varepsilon \le \underline{\lim} F_{V_n}(t) \le \overline{\lim} F_{V_n}(t) \le F_U(t+\varepsilon) + \varepsilon$$

$\lim_{n\to\infty} F_{U_n}(t-\varepsilon) = F_U(t-\varepsilon)$ if $t-\varepsilon$, $t+\varepsilon$ are continuity pt of F_U & let $\varepsilon \to 0$

$$\therefore \underline{\lim} F_{V_n}(t) = \overline{\lim} F_{V_n}(t) = \lim_{n\to\infty} F_{U_n}(t) = F_U(t)$$

$$\lim_{n\to\infty} F_{V_n} = F_U \quad \forall\, t \in \text{continuity set}$$

$$\therefore U_n \xrightarrow{d} U$$



Step 2 : $X_n \xrightarrow{d} X$, $Y_n \xrightarrow{P} 0$ $\Longrightarrow$ $X_n \cdot Y_n \xrightarrow{P} 0$

$$\underbrace{X_n \cdot Y_n}\, - \underbrace{a X_n} = X_n (Y_n - a) \qquad \{ X_n Y_n \equiv U_n \ \& \ a X_n \equiv V_n \}$$

$$X_n \xrightarrow{d} X , \quad Y_n - a \xrightarrow{P} 0 \qquad [X_n \xrightarrow{P} X , \ Y_n \to a]$$

$$\left. \begin{array}{l} X_n \cdot Y_n - a X_n \xrightarrow{P} 0 \\ a X_n \xrightarrow{d} a X \end{array} \right\} \Longrightarrow X_n Y_n \xrightarrow{d} a X$$

Complete proof will be given by Yinshu

Example CLT is an application of this

Variable Transformation

Proof of Slutsky's theorem will not be asked for. But results used in the proof - Step 1, 2, 3 you must know.

$X_n \xrightarrow{d} X$

$$\lim_{n \to \infty} F_{X_n}(t) = F_X(t)$$

$$\lim_{n \to \infty} P(X_n \le t) = P(X \le t)$$

$$\lim_{n \to \infty} P(a X_n \le t) = \lim P(X_n \le t/a)$$

$$= \lim P(X \le t/a) \qquad t/a \text{ is a cont}$$

$$= P(aX \le t)$$

(2)

$$= \lim_{m\to\infty} P\left(\bigcup_{k=m}^{\infty} A_K\right) \qquad \left[\because \left(\bigcup_{k=m}^{\infty} A_K\right) \text{ is } \downarrow \text{ \& meas is finite}\right]$$

$$\leq \lim_{m\to\infty} \sum_{k=m}^{\infty} P(A_K) \qquad [\text{~~finite~~ Countable Subadditivity}]$$

$$= \lim_{m\to\infty} \sum_{k=m}^{\infty} \frac{1}{2^k}$$

$$P(c^c) = \lim_{m\to\infty} \frac{1}{2^{m-1}} = 0$$

To show $\{X_{n_K}\}$ is Cauchy $\forall\, w \in C$

$$w \in C \Rightarrow w \in B_{m_0}^c \quad \text{for some } m_0$$

$$\Rightarrow w \in \bigcap_{k=m_0}^{\infty} A_K^c$$

$$\Rightarrow w \in A_K^c \qquad \forall\, k \geq m_0$$

$$\Rightarrow w \in \left\{ |X_{n_K} - X_{n_{K+1}}| > \tfrac{1}{2^K} \right\}^c \qquad \forall\, k \geq m_0$$

$$\Rightarrow |X_{n_K}(w) - X_{n_{K+1}}(w)| \leq \frac{1}{2^K} \qquad \forall\, k \geq m_0$$

For $n_i > n_j \geq m$

$$|X_{n_i} - X_{n_j}| = |X_{n_i} - X_{n_{i+1}} + X_{n_{i+1}} - \cdots\cdots X_{n_j}|$$

$$\leq |X_{n_i} - X_{n_{i+1}}| + |X_{n_{i+1}} - X_{n_{i+2}}| + \cdots$$

$$= \sum_{k=i}^{\infty} |X_{n_K} - X_{n_{K+1}}|$$

$$\leq \sum_{k=i}^{\infty} \frac{1}{2^K} = \frac{1}{2^{i-1}}$$

$$\therefore |X_{n_i} - X_{n_j}| < \frac{1}{2^{i-1}}$$

$$\Rightarrow |X_{n_i} - X_{n_j}| < \epsilon' \qquad \epsilon' = \frac{1}{2^{i-1}}$$

$\therefore$ $\{X_{n_K}\}$ is a.e. Cauchy

Now, $X = \lim_{n\to\infty} X_{n_K}$ $\forall\, w \in C$ & $\lim_{n\to\infty} X_{n_k} = 0$ ow

Finally, $|X_n - X| \leq |X_n - X_{n_K}| + |X_{n_K} - X|$

$$P\{|X_n - X| > \epsilon\} \leq \underbrace{P\{|X_n - X_{n_K}| > \epsilon/2\}}_{\downarrow 0 \text{ Cauchy}} + \underbrace{P\{|X_{n_K} - X| > \epsilon/2\}}_{\downarrow 0}$$

$X_{n_K} \to X$ a.e.

$\mu(\Omega) < \infty$

$\Downarrow$

$X_{n_k} \xrightarrow{P} X$

$$\therefore P\{|X_n - X| > \epsilon\} \Rightarrow 0$$

$$\therefore X_n \xrightarrow[\mu]{} X. \qquad \text{q.e.d.}$$

Prop 4.1 : $X_n \xrightarrow{p} X \Rightarrow X_n \xrightarrow{d} X$ $\quad\therefore\quad X_n \xrightarrow{p} c \Rightarrow X_n \xrightarrow{d} c$

① Only need to show that given $X_n \xrightarrow{d} c \Rightarrow X_n \xrightarrow{p} c$ $\quad (P(|X_n - c| > \epsilon) \underset{n \to \infty}{\longrightarrow} 0)$

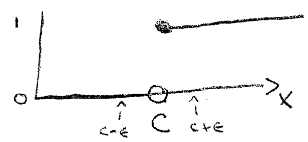
② $P(|X_n - c| > \epsilon) = P(X_n - c > \epsilon) \cup P(X_n - c < -\epsilon) = P(X_n - c > \epsilon) + P(X_n - c < \epsilon)$ $\because$ disjoint

$= P(X_n > c + \epsilon) + P(X_n < c - \epsilon) = [1 - P(X_n \leq c + \epsilon)] + P(X_n < c - \epsilon)$

$\leq 1 - P(X_n \leq c + \epsilon) + P(X_n \leq c - \epsilon)$ $\quad\because\quad P(f(x) < c) \subset P(f(x) \leq c)$

$\therefore P(|X_n - c| > \epsilon) \leq 1 - F_{X_n}(c + \epsilon) + F_{X_n}(c - \epsilon)$ $\quad\because\quad P(X \leq x) = F_X(x)$

③ Consider $\underbrace{X = c}_{\text{no randomness}}$, $F_X(x) = F_c(x) = P(c \leq x) = \begin{cases} 1 & x \geq c \\ 0 & \text{o.w} \end{cases}$



④ Recall $X_n \xrightarrow{d} c \iff F_{X_n}(x) \to F_X(x)$ at each continuity point of F

Only one pt. of discontinuity at c

$\therefore$ $F_{X_n}(c + \epsilon) \to F_X(c + \epsilon) = 1$ $\quad\because\quad (c + \epsilon)$ + $(c - \epsilon)$ are continuity pts of F

$F_{X_n}(c - \epsilon) \to F_X(c - \epsilon) = 0$

⑤ $\underline{\lim} P(|X_n - c| > \epsilon) \leq 1 - \lim F_{X_n}(c + \epsilon) + \lim F_{X_n}(c - \epsilon) = 1 - 1 + 0 = 0$ $\left\{ \begin{array}{l} \because \lim F_{X_n}\text{'s exists} \\ \underline{\lim} = \lim \end{array} \right.$

$\therefore 0 \leq \underline{\lim}\, P(|X_n - c| > \epsilon) \leq 0 \quad \to \quad \underline{\lim}\, P(|X_n - c| > \varepsilon) = 0$

Similarly $0 \leq \overline{\lim}\, P(|X_n - c| > \epsilon) \leq 0 \to \overline{\lim}\, P(|X_n - c| > \varepsilon) = 0$.

$\therefore \lim P(|X_n - c| > \epsilon) = 0$ $\quad\because\quad \underline{\lim} = \overline{\lim} \Rightarrow \lim$ exists

$\therefore X_n \xrightarrow{p} X = c$

Problem 1 $\qquad X_n \xrightarrow{d} a \iff X_n \xrightarrow{p} a$