Asymptotics - Casella & Lehmann
- Sen & Singer

LSI - CR Rao.

SLUTSKY's THEOREM: If $X_n \xrightarrow{d} X$ & $Y_n \xrightarrow{P} a$, $Z_n \xrightarrow{P} b$ th

$X_n \cdot Y_n + Z_n \xrightarrow{d} aX+b$

Proof

Step 1: $X_n \xrightarrow{d} X$ & $Y_n \xrightarrow{P} a \Rightarrow X_n \cdot Y_n \xrightarrow{d} aX$

Step 2: If $M_n \xrightarrow{d} M$ & $L_n \xrightarrow{P} b \Longrightarrow L_n + M_n \xrightarrow{d} M+b$

Step 3: If $M_n \xrightarrow{d} M \Rightarrow M_n + b \xrightarrow{d} M+b$

These together $\Rightarrow X_n Y_n + Z_n \xrightarrow{d} aX+b$

Step 4: If $U_n - V_n \xrightarrow{P} 0$ & $U_n \xrightarrow{d} U \Longrightarrow V_n \xrightarrow{d} U$

Prove 4: Let $U_n = X_n Y_n + b$ & $V_n = X_n Y_n + Z_n$

Using ① & ③ we know $U_n \xrightarrow{d} aX+b$

We also know $\underbrace{U_n - V_n}_{Z_n - b} \xrightarrow{P} 0$

$\Rightarrow V_n \xrightarrow{d} a$

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If $X_n \xrightarrow{d} a \Longleftrightarrow X_n \xrightarrow{P} a$

Proof We know $X_n \xrightarrow{P} X \Longrightarrow X_n \xrightarrow{d} X$ {Result proved in

$\therefore X_n \xrightarrow{P} a \Longrightarrow X_n \xrightarrow{d} a$

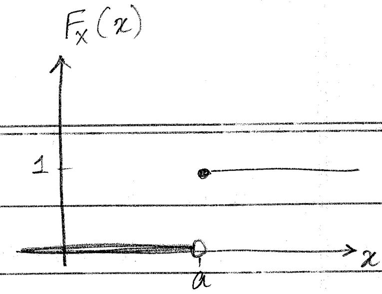
To show given $X_n \xrightarrow{d} a \Longrightarrow X_n \xrightarrow{P} a$

ie to show $\forall \varepsilon > 0 \quad P(|X_n - a| > \varepsilon) \longrightarrow 0$ as $n \to \infty$

Now

$$\begin{aligned} P(|X_n-a|>\varepsilon) &= P(X_n - a > \varepsilon) + P(X_n - a < -\varepsilon) \\ &= P(X_n > a+\varepsilon) + P(X_n < a-\varepsilon) \\ &= [1 - P(X_n \le a+\varepsilon)] + P(X_n < a-\varepsilon) \\ &\le 1 - P(X_n \le a+\varepsilon) + P(X_n \le a-\varepsilon) \end{aligned}$$

$$P(|X_n-a|>\varepsilon) \le 1 - F_{X_n}(a+\varepsilon) + F_{X_n}(a-\varepsilon).$$



Now for $X=a$ $\quad F_X(x) = \begin{cases} 1 & x \geq a \\ 0 & \text{ow} \end{cases}$

Now $X_n \xrightarrow{d} a \iff F_{X_n}(x) \longrightarrow F_X(x) \quad \forall \; x \in C(F_X)$ continuity pt

The funct^{n} F_X has only one pt of discontinuity namely 'a'.

Now $\underline{\lim}\, P(|X_n - X| > \varepsilon) \leq \underline{\underline{\lim}} \; 1 - F_{X_n}(a+\varepsilon) + F_{X_n}(a-\varepsilon)$

Also $\quad F_{X_n}(a+\varepsilon) \longrightarrow F_X(a+\varepsilon) = 1$
$\quad\quad\;\; F_{X_n}(a-\varepsilon) \longrightarrow F_X(a-\varepsilon) = 0$

{ since $(a+\varepsilon)$ & $(a-\varepsilon)$ are continuity pts of F

$\therefore \underline{\lim}\; P(|X_n - X| > \varepsilon) \leq 1 - \lim F_{X_n}(a+\varepsilon) + \lim F_{X_n}(a-\varepsilon)$ { since lim exists

$$= 1 - 1 + 0$$
$$= 0$$

$\therefore \; 0 \leq \underline{\lim}\; P(|X_n - X| > \varepsilon) \leq 0$ $\qquad$ { $\underline{X=a}$

$|||^{rly}$ $\quad 0 \leq \overline{\lim}\; P(|X_n - X| > \varepsilon) \leq 0.$

$\therefore \; \lim P(|X_n - X| > \varepsilon) = 0$

$\therefore \; X_n \xrightarrow{P} X$

<u>NOTE:</u> $X_n \xrightarrow{d} X, \; Y_n \xrightarrow{d} a, \; Z_n \xrightarrow{d} b \Rightarrow X_n Y_n + Z_n \xrightarrow{d} aX + b$

Slutsky's theorem & above result combined give this.

#3] $X_m - X_n \xrightarrow{\mu} 0 \iff X_n \xrightarrow{\mu} X$

ie If $(X_n - X_m) \xrightarrow{\mu} 0$ then $\exists$ an X st $X_n \xrightarrow{\mu} X$

& If $X_n \xrightarrow{\mu} X$ then $(X_n - X_m) \xrightarrow{\mu} 0$

<u>Step 1</u>: Given $X_n \xrightarrow{\mu} X \Rightarrow P(|X_n - X_m| > \varepsilon) \longrightarrow 0$ as $m \wedge$

$X_n \xrightarrow{\mu} X \Rightarrow P(|X_n - X| > \varepsilon) \longrightarrow 0$.

Consider $P(|X_m - X_n| > \varepsilon) \Rightarrow P(|X_m - X + X - X_n| > \varepsilon)$

$\{|X_m - X| \leq \varepsilon/2\} \cap \{|X_n - X| \leq \varepsilon/2\} \subseteq \{|X_m - X_n| \leq \varepsilon\}$

$\therefore \{|X_m - X| > \varepsilon/2\} \cup \{|X_n - X| > \varepsilon/2\} \supseteq \{|X_m - X_n| > \varepsilon\}$

$\therefore P[(|X_m - X| > \varepsilon/2) \cup (|X_n - X| > \varepsilon/2)] \geq P(|X_m - X_n| > \varepsilon)$

$\therefore P(|X_m - X| > \varepsilon/2) + P(|X_n - X| > \varepsilon/2) \geq P(|X_m - X_n| > \varepsilon)$

$X_n \xrightarrow{\mu} X$ so

$\exists N_1 \& N_2$ st $\forall m \geq N_1 \quad P(|X_m - X| > \varepsilon/2) \leq \delta \qquad \forall \varepsilon > 0$

$\forall n \geq N_2 \quad P(|X_n - X| > \varepsilon/2) < \delta$

$\therefore P(|X_m - X_n| > \varepsilon) \leq 2\delta$ for $(m \wedge n) \geq \max\{N_1, N_2$

<u>Step 2</u> Given $X_m - X_n \xrightarrow{\mu} 0$

1. Extract $\{X_{n_k}\}$ from $\{X_n\}$
2. Show $\{X_{n_k}\}$ is a.e Cauchy → Define C st $\forall \omega \in C$, $\{X_{n_k}(\omega$ ca… ; → $P(C^c) = 0$
3. $\lim \{X_{n_k}\}$ exists a.e
4. Define $X = \begin{cases} \lim_k X_{n_k} & \text{where lim exists} \quad \omega \in C \\ = 0 & \text{OW} \quad \omega \notin C \end{cases}$
5. Show $X_n \xrightarrow{\mu} X$

① $X_m - X_n \xrightarrow{\mu} 0 \Rightarrow \forall\, \varepsilon > 0 \quad \mu(|X_m - X_n| > \varepsilon) \leq \varepsilon \quad \forall \quad n \geq N_\varepsilon$

$\therefore \quad P\left(|X_n - X_m| > 1/2^k\right) < 1/2^k \quad \forall \quad n \geq N_k$

Now $\exists \{n_k\}_{k\geq 1}$ st $P\left(|X_{n_k} - X_m| > 1/2^k\right) < 1/2^k \quad \forall \quad m \geq n_k \quad \because$

For $k=1$, n_1 then $P\left(|X_m - X_{n_1}| > 1/2^k\right) < 1/2^k \quad \forall \quad m \geq n_1$

For $k=2$ we can find n_2 st $P\left(|X_m - X_{n_2}| > 1/2^k\right) < 1/2^k \quad \forall\, m \geq n_1$

Note $n_1 \geq N(1)$, $n_2 \geq \max(N(2), n_1)$

② Define $A_k = \bigcup_{l \geq k} \left[|X_{n_k} - X_l| > 1/2^k\right]$

$$B_m = \bigcup_{k=m}^{\infty} A_k$$

Let $C = \bigcup_{m=1}^{\infty} B_m^c$

check if $\underline{\lim}$ or $\overline{\lim}$

Now $P(C^c) = P\left(\bigcap_{m=1}^{\infty} B_m\right) \leq \overline{\lim}_{m\to\infty} P(B_m) \leq \lim_{m\to\infty} (1/2^{m-1}) = 0$

$\left[\because \quad P(B_m) \leq \sum_{k=m}^{\infty} P(A_k) \leq \sum_{k=m}^{\infty} (1/2^k) = 1/2^{m-1}\right]$

To show $\{X_{n_k}\}$ is Cauchy $\forall\, \omega \in C$.

$\omega \in C \Rightarrow \omega \in B_{m_0}^c$ ie $\omega \in \bigcap_{k \geq m_0} A_k^c$ for some m_0

$\Rightarrow \omega \in A_k^c \quad \forall\, k \geq m_0$

$\Rightarrow \omega \in \left(\bigcup_{l \geq k} |X_{n_k} - X_l| > 1/2^k\right)^c \quad \forall\, k \geq m_0$

? $\Rightarrow |X_{n_k}(\omega) - X_l(\omega)| < 1/2^k \quad \forall\, k \geq m_0$

$$|X_{n_i} - X_{n_j}| = |X_{n_i} - X_{n_{i+1}} + X_{n_{i+1}} - X_{n_{i+2}} - X_{n_{i+2}} + \dots - X_{n_j}|$$

$$\leq \sum_{k=n_i}^{\infty} |X_k - X_{k+1}|$$

$$\leq \sum_{k=n_i}^{\infty} 1/2^k = 1/2^{n_i - 1}$$

$|X_{n_i} - X_{n_j}| < 1/2^{n_i-1}$ $\Rightarrow \{X_{n_k}\}$ is almost everywhere cauchy

(3) Let $X = \begin{cases} \lim_{k\to\infty} \{X_{n_k}\} & \forall\ w \in C \\ 0 & \text{o/w} \end{cases}$ [If it is cauchy limit exists.

(4) To show $X_n \xrightarrow{\mu} X$ ie $\overset{P}{\mu}(|X_n - X| > \varepsilon) \longrightarrow 0$ [μ must be finite

$$|X_n - X| \leq |X_n - X_{n_k}| + |X_{n_k} - X|$$

$$P(|X_n - X| > \varepsilon) \leq \underbrace{P(|X_n - X_{n_k}| > \varepsilon)}_{\downarrow \text{ cauchy } 0} + \underbrace{P(|X_{n_k} - X| > \varepsilon)}_{\downarrow\ 0}$$

since $X_{n_k} \to X$ a. for $\mu(\Omega) < \infty$

$\therefore P(|X_n - X| > \varepsilon) \longrightarrow 0$

prop done in c