Reading : Pg 46 - Inequalities from T. Bk.

Pg 48 Ex 4.2 } Extra Credit
Pg 50 Ex 4.4 }

Homework 4 - STA 5446

Due Tue, Nov the 8^{th} 2005

1. Problem 3.2.1 page 43. If $E(X)=0$ & $X \geq 0 \Rightarrow X=0$ a.e.
2. Problem 3.2.2 page 43. If $E(X)=0$ does not mean $X=0$ always since it has to be $\int$ over all $A \in \mathcal{A}$ not just over Ω.
3. Problem 3.2.3 page 43.
4. Problem 3.4.3 page 48
(Hint: Use Hölder's inequality).
5. Suppose that $\varepsilon_1, \dots, \varepsilon_n$ are i.i.d r.v.'s such that $P(\varepsilon_i = 1) = P(\varepsilon_i = -1) = \frac{1}{2}$, for all i.
Let $a_i \in \mathbb{R}$, for $i = 1, \dots, n$.
Show that:
$$A \left(\sum_{i=1}^{n} a_i^2 \right)^{1/2} \leq E \left| \sum_{i=1}^{n} a_i \varepsilon_i \right| \leq B \left(\sum_{i=1}^{n} a_i^2 \right)^{1/2},$$
for some constants A and B. (indep of n)

Used in Empirical Processes

Hint: Use Problem 4.

Suppose $a_i \geq 0$ then if you toss a coin & you see a $H = +1$ you gain a_i, $E|\sum a_i \varepsilon_i|$ = expected gain and you get upper & lower bdds on this expected gain

The prop $\quad X : (\Omega, \mathcal{A}) \longrightarrow (\mathbb{R}, \mathcal{B}) \qquad \mathcal{A}_0$ is a sub σ-field of $\mathcal{A}$.

$$\int X \cdot d\mu = \int X \cdot d\mu_0 \qquad \forall\ A \in \mathcal{A}_0$$

where $\quad \mu_0 \equiv \mu \,|_{\mathcal{A}_0}$

ets used

If μ = lebesgue meas on $\mathbb{R}$ $= \lambda$

but if we want to work with the lebesgue meas restricted to $[0,1]$ ie $\lambda_{[0,1]}$

Holders Ineq ⁂
Cauchy schwarz $\qquad E(|xy|) \leq [E(x^2)]^{1/2} \, [E(y^2)]^{1/2}$

If you want to show $E(|xy|) < \infty$ it might be easier to look at $E(x^2)$ & $E(y^2)$

$$\boxed{X = \overset{X^+}{C_1} - \overset{X^-}{C_2} \quad \text{but} \quad \int X^+ = \infty, \quad \int X^- = \infty}$$

If $\mu = P$ prob measure & $y = 1$ the cauchy Schwarz gives

$$E|X| \leq [E(X^2)]^{1/2}$$

since $E(1) = \int d\mu = 1$
since μ is prob meas

#1] Show that if $X \geq 0$ and $\int X \cdot d\mu = 0 \Rightarrow \mu([X>0]) = 0$ ie $\{X = 0 \text{ a.e}\}$

Proof We will show $\mu([X>0]) > 0 \Rightarrow \int X \cdot d\mu > 0$ (proof by contradiction)

Let if possible $\mu([X>0]) > 0$ $\{\mu([X>0]) \geq 0 \text{ always}\}$

$$[X>0] = \bigcup_{n=1}^{\infty} \{\omega \in \Omega : X(\omega) > 1/n\} \equiv \bigcup_{n=1}^{\infty} A_n \quad \{\text{where } A_n = \{\omega : X(\omega) > 1/n\}\}$$

$$\mu([X>0]) = \mu\left(\bigcup_{n=1}^{\infty} \{\omega \in \Omega : X(\omega) > 1/n\}\right) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right)$$

$$= \lim_{n\to\infty} \mu(A_n) > 0 \quad \left\{ \begin{array}{l} A_n \uparrow \text{ seq} \\ \therefore \mu(\bigcup_n A_n) = \lim \mu(A_n) \end{array} \right.$$

Since $\lim_{n\to\infty} \mu(A_n) > 0 \Rightarrow \exists$ some n_0 st $\mu(A_{n_0}) > 0$ —— (*)

Now $\int X \cdot d\mu \geq \int_{A_{n_0}} X \, d\mu$ $\{A_{n_0} = \{\omega \in \Omega : X(\omega) > 1/n_0\}\}$

$$\geq \int_{A_{n_0}} (1/n_0) \, d\mu = \int (1/n_0) \, I_{A_{n_0}} \cdot d\mu$$

$\therefore \int X \cdot d\mu \geq (1/n_0)\, \mu(A_{n_0}) > 0$ $\{\because \frac{1}{n_0} > 0$ & $\mu(A_{n_0}) > 0\}$

$\therefore \int X \, d\mu > 0$

we have a contradiction

#2]

$$\int_A X \cdot d\mu = \begin{cases} = 0 \\ \geq 0 \end{cases} \forall A \in \mathcal{A} \Rightarrow X = \begin{cases} = 0 & \text{a.e} \\ \geq 0 & \text{a.e} \end{cases}$$

Proof Step 1 given $\int_A X \cdot d\mu = 0$ we have to show $X = 0$ a.e

Since $\int_A X \cdot d\mu$ is defined, X is measurable on $\mathcal{A}$ $[X : (\Omega, \mathcal{A}) \longrightarrow (\mathbb{R}, \mathcal{B})]$

Now $X^{-1}([0,\infty)) \subset \mathcal{A} \Leftrightarrow \{\omega \in \Omega : X(\omega) > 0\} = [X>0] \in \mathcal{A}$

Let $A = [X>0]$ then we have $\int_A X \cdot d\mu = 0 \Rightarrow \int X\, I_{[X>0]}\, d\mu = 0$

$$\Rightarrow \int X^{+} \cdot d\mu = 0$$

$$\Rightarrow X^{+} = 0 \quad \text{a.e} \quad \left\{ \begin{array}{l} \because X^{+} \geq 0 \ \& \int X^{+} = 0 \\ \Rightarrow X^{+} = 0 \quad \text{by Q1} \end{array} \right.$$

Also $X^{-1}((-\infty,0]) \subset \mathcal{A} \Rightarrow \{\omega \in \Omega : X(\omega) < 0\} = [X<0] \in \mathcal{A}$ [$\because$ by meas of X

Let $A = [X<0]$ then we have $\int_A X \cdot d\mu = 0 \Rightarrow \int X\, I_{[X<0]} = 0$

$$\Rightarrow \int -(X^{-}) \cdot d\mu = 0$$

$$\Rightarrow \int X^{-} \cdot d\mu = 0$$

$$\Rightarrow X^{-} = 0 \quad \text{a.e}$$

$\therefore$ we have $X^{+} = 0$ a.e and $X^{-} = 0$ a.e $\Rightarrow \mu([X^{+} \neq 0]) = 0$ & $\mu([X^{-} \neq 0]) = 0$

Now $0 \leq \mu([X \neq 0]) = \mu(\{\omega \in \Omega : X(\omega) \neq 0\})$

$$\leq \mu([X^{+} \neq 0] \cup [X^{-} \neq 0]) \quad \{ [X \neq 0] \subseteq [X^{+} \neq 0] \cup [X^{-} \neq 0]$$

$$\leq \mu([X^{+} \neq 0]) + \mu([X^{-} \neq 0])$$

$\therefore \ 0 \leq \mu([X \neq 0]) \leq 0$

ie $\mu([X \neq 0]) = 0 \Rightarrow X = 0$ a.e

Step 2 given $\int_A X \cdot d\mu \geq 0$ for any $A \in \mathcal{A} \Rightarrow X \geq 0$ a.e

Now to show $X \geq 0$ a.e $\Leftrightarrow X^{+} - X^{-} \geq 0$ a.e

$$\Leftrightarrow \mu([X^{+} - X^{-} < 0]) = 0$$

$$\Leftrightarrow \mu\left(\left[X\, I_{[X>0]} < -X\, I_{[X<0]}\right]\right) = 0 \quad \left\{ \begin{array}{l} \text{by def of} \\ X^{+} \ \& \ X^{-} \end{array} \right.$$

$$\Leftrightarrow \mu([X<0]) = 0 \quad \left\{ \begin{array}{l} \because \{\omega : X^{+}(\omega) < X^{-}(\omega)\} \\ \quad \shortparallel \\ \{\omega : X\, I(X(\omega)>0) < -X\, I(X(\omega)<0)\} \\ \quad \shortparallel \\ \{\omega : X(\omega) < 0\} \end{array} \right.$$

$\therefore$ we need to show $X^{-} = 0$ a.e $\quad$ { since $X^{-} \geq 0$ always

Let $A = \{ w \in \Omega \mid X(\omega) < 0 \}$

If $\int_A X \cdot d\mu \geq 0 \Rightarrow \int X \, I_{[X<0]} \, d\mu \geq 0$

$\Rightarrow \int -(X^-) \cdot d\mu \geq 0$ $\quad \left\{ \because \text{ by def } X^- = \begin{cases} -X & \text{if } X<0 \\ 0 & \text{ow} \end{cases} \right.$

$\Rightarrow \int X^- \, d\mu \leq 0$ ——(**)

Now $X^- \geq 0$ always $\Rightarrow \int X^- \cdot d\mu \geq 0$ ——(***)

$\therefore$ by (**) & (***) $\Rightarrow \int X^- d\mu = 0 \Rightarrow X^- = 0$ a.e $\{ \because$ by step 1

$\Rightarrow X = X^+$ a.e $\quad \{ \because X = X^+ - X^-$

$\Rightarrow X \geq 0$ a.e $\quad \{ \because X^+ \geq 0$ always

\# 3] $(\Omega, \mathcal{A}, \mu)$ is a measure space. $\mathcal{A}_0$ is a sub σ-field of $\mathcal{A}$

$\mu_0 \equiv \mu\big|_{\mathcal{A}_0}$ $\quad$ [ie $\mu_0(A) = \mu(A) \;\; \forall A \in \mathcal{A}_0$ ie they agree on $\mathcal{A}_0 \subset \mathcal{A}$]

To show $\int X \cdot d\mu = \int X \cdot d\mu_0$ for any X which is $\mathcal{B}-\mathcal{A}_0$ measurable.

Proof Notice X is $\mathcal{B}-\mathcal{A}_0$ meas $\rightarrow X$ is also $\mathcal{B}-\mathcal{A}$ meas

since X is $\mathcal{B}-\mathcal{A}_0$ meas $\Rightarrow X^{-1}(\mathcal{B}) \subset \mathcal{A}_0 \subset \mathcal{A} \Rightarrow X^{-1}(\mathcal{B}) \subset \mathcal{A}$

$\rightarrow X$ is $\mathcal{B}-\mathcal{A}$ meas.

Step 1 Let X be a simple funct$^{\underline{n}}$. $X = \sum_{i=1}^{m} x_i I_{A_i}$ where $x_i \geq 0$ & $\sum A_i = \Omega$

$\int I_{A_i} \, d\mu = \mu(A_i) \rightarrow \int x_i I_{A_i} d\mu = x_i \mu(A_i)$

$\int I_{A_i} \, d\mu_0 = \mu_0(A_i) = \mu(A_i) \rightarrow \int x_i I_{A_i} \, d\mu_0 = x_i \mu(A_i)$

$\Rightarrow \int x_i I_{A_i} d\mu = \int x_i I_{A_i} d\mu_0$

$\therefore \int X \, d\mu = \int \sum x_i I_{A_i} \, d\mu = \sum_{i=1}^{n} \int x_i I_{A_i} \, d\mu = \sum_{i=1}^{n} \int x_i I_{A_i} d\mu_0$

$= \int \sum x_i I_{A_i} \, d\mu_0$

$= \int X \cdot d\mu_0$

Step 2 Let $X \geq 0$ then $\exists \{f_n\}$ seq of ≥ 0 simple functⁿ st $f_n \uparrow X$

ie $\lim_{n \to \infty} f_n(\omega) = X(\omega)$

Now $\int X \cdot d\mu = \int \lim_{n \to \infty} f_n \, d\mu$

$= \lim_{n \to \infty} \int f_n \, d\mu$ $\{\because$ by M.C.T. $f_n \uparrow X$ & $f_n \geq 0$ $\Rightarrow \int f_n \uparrow \int X$

$= \lim_{n \to \infty} \int f_n \, d\mu_0$ $\{$ f_n is seq of simple functⁿ so by step 1

$= \int \lim_{n \to \infty} f_n \cdot d\mu_0$ $\{$ by MCT

$= \int X \, d\mu_0$

Step 3 Let X be any arbitrary measurable functⁿ

We can write $X = X^+ - X^-$

Now $\int X^+ d\mu = \int X^+ d\mu_0$ $\{\because$ $X^+ \geq 0$ & $X^- \geq 0$ using step 2

$\int X^- d\mu = \int X^- d\mu_0$

[If at least one of $\int X^+ d\mu$ or $\int X^- d\mu$ are finite the $\int X^+ d\mu - \int X^- d\mu$ is defined (should be a condition stated in the question.)]

$\int X^+ d\mu - \int X^- d\mu = \int X^+ d\mu_0 - \int X^- d\mu_0$

ie $\int (X^+ - X^-) \, d\mu = \int (X^+ - X^-) \, d\mu_0$ $\{$ $\int X^+ - X^- d\mu = \int X^+ d\mu - \int X^- d\mu$

ie $\int X \cdot d\mu = \int X \cdot d\mu_0$ [as long as $\int X^+ d\mu$ or $\int X^- d\mu < \infty$]

#4] To show $m_r^{s-t}\, m_t^{r-s} \geq m_s^{r-t}$ $\qquad r \geq s \geq t \geq 0$

where $m_r = E(|X|^r)$

$\therefore$ to show $\{E(|X|^r)\}^{s-t}\{E(|X|^t)\}^{r-s} \geq \{E(|X|^s)\}^{r-t}$

$$\Leftrightarrow \quad \{E(|X|^r)\}^{\frac{s-t}{r-t}}\{E(|X|^t)\}^{\frac{r-s}{r-t}} \geq \{E(|X|^s)\}$$

$$\Leftrightarrow \quad \{E(|X|^r)\}^{1/\left(\frac{s-t}{r-t}\right)}\{E(|X|^t)\}^{1/\left(\frac{r-t}{r-s}\right)} \geq \{E(|X|^s)\}$$

$$\Leftrightarrow \quad \left\{E\left(|X|^{r\left(\frac{r-t}{s-t}\right)\left(\frac{s-t}{r-t}\right)}\right)\right\}^{1/\left(\frac{s-t}{r-t}\right)}\left\{E\left(|X|^{t\left(\frac{r-t}{r-s}\right)\left(\frac{r-s}{r-t}\right)}\right)\right\}^{1/\left(\frac{r-t}{r-s}\right)} \geq E(|X|^s)$$

$$\Leftrightarrow \quad \left\{E\left[\left(|X|^{r\frac{(s-t)}{(r-t)}}\right)^{\left(\frac{r-t}{s-t}\right)}\right]\right\}^{1/\left(\frac{r-t}{s-t}\right)}\left\{E\left[\left(|X|^{t\frac{(r-s)}{(r-t)}}\right)^{\left(\frac{r-t}{r-s}\right)}\right]\right\}^{1/\left(\frac{r-t}{r-s}\right)} \geq E(|X|^s)$$

Now Holders inequality says $E|XY| \leq \{E(|X|^r)\}^{1/r}\{E(|Y|^s)\}^{1/s}$; $r>1$ & $\left(\frac{1}{r}+\frac{1}{s}\right)=1$

In Holders inequality $E|MN| \leq [E(|M|^p)]^{1/p}[E(|N|^q)]^{1/q}$; $p>1$ & $\frac{1}{p}+\frac{1}{q}=1$

let
$M = |X|^{\left(\frac{rs-rt}{r-t}\right)}$

$N = |X|^{\left(\frac{rt-st}{r-t}\right)}$

$p = \frac{r-t}{s-t}$

$q = \frac{r-t}{r-s}$

$\Rightarrow$

$MN = |X|^{\frac{rs-rt+rt-st}{r-t}} = |X|^s$

$\frac{1}{p}+\frac{1}{q} = \frac{s-t}{r-t}+\frac{r-s}{r-t} = \frac{r-t}{r-t} = 1$

$\because r \geq s \rightarrow r-t \geq s-t \rightarrow \frac{r-t}{s-t} \geq 1$

$\therefore$ applying Holders inequality the result follows.

If $r=4,\ s=2,\ t=1 \Rightarrow m_4\, m_1^2 \geq m_2^3$

5] Given $\varepsilon_1, \varepsilon_2 \cdots \varepsilon_n$ are iid r.v st $P(\varepsilon_i = 1) = P(\varepsilon_i = -1) = 1/2 \quad \forall i$. Let $a_i \in \mathbb{R}$,

To show $A\left(\sum_{i=1}^{n} a_i^2\right)^{1/2} \leq E\left(\left|\sum_{i=1}^{n} a_i \varepsilon_i\right|\right) \leq B\left(\sum_{i=1}^{n} a_i^2\right)^{1/2}$

Proof 1) Liapunov's inequality says $\{E(|X|^r)\}^{1/r}$ is $\uparrow$ in r (if $\mu(\Omega) < \infty$)

So $\{E(|X|)\} \leq \{E(|X|^2)\}^{1/2}$ ——— $(*)$

Let
$$X = \sum_{i=1}^{n} a_i \varepsilon_i$$
$$X^2 = \sum_{i=1}^{n} a_i^2 \varepsilon_i^2 + \sum_{i \neq j = 1}^{n} a_i a_j \varepsilon_j \varepsilon_i$$
$$E(X^2) = \sum_{i=1}^{n} a_i^2 E(\varepsilon_i^2) + \sum_{i \neq j}^{n} a_i a_j E(\varepsilon_i \varepsilon_j)$$
$$= \sum_{i=1}^{n} a_i^2 (1) + 0$$
$\left\{ \begin{array}{l} \because E(\varepsilon_i) = 0 \\ V(\varepsilon_i) = E(\varepsilon_i^2) = 1 \end{array} \right.$

$\therefore$ we get $E\left(\left|\sum_{i=1}^{n} a_i \varepsilon_i\right|\right) \leq \left(\sum_{i=1}^{n} a_i^2\right)^{1/2}$ ——— I $\quad$ $\left\{ \begin{array}{l} \text{applying } (*) \\ E(|X|^2) = E(X^2) \end{array} \right.$

2) Now from #4 we have $\{E(|X|^4)\}\{E|X|\}^2 \geq \{E|X|^2\}^3$

ie
$$\{E|X|\} \geq \frac{\{E(|X|^2)\}^{3/2}}{\{E(|X|^4)\}^{1/2}}$$

$$\therefore \quad E\left(\left|\sum_{i=1}^{n} a_i \varepsilon_i\right|\right) \geq \frac{\left\{\sum_{i=1}^{n} a_i^2\right\}^{3/2}}{\{E(|X|^4)\}^{1/2}} \quad \text{——— } (*)$$

$\therefore$ If we can show $\dfrac{1}{\{E(|X|^4)\}^{1/2}} \geq \dfrac{D}{\left\{\sum_{i=1}^{n} a_i^2\right\}}$ ——— $(**)$ $\quad$ [then we are done from $(*)$ & $(**)$]

$\therefore$ we will show $\left\{\dfrac{1}{D}\sum_{i=1}^{n} a_i^2\right\}^2 \geq \{E(|X|^4)\}$

Now
$$E(|X|^4) = E(X^4) = E\left[\left(\sum_{i=1}^{n} a_i \varepsilon_i\right)^4\right]$$
$$= E\left[\sum_{i}\sum_{j}\sum_{k}\sum_{l} a_i a_j a_k a_l \, \varepsilon_i \varepsilon_j \varepsilon_k \varepsilon_l\right]$$

$$E(|X|^4) = \sum_i \sum_j \sum_k \sum_l a_i a_j a_k a_l \, E(\varepsilon_i \varepsilon_j \varepsilon_k \varepsilon_l)$$

Now $E(\varepsilon_i \varepsilon_j \varepsilon_k \varepsilon_l)$ is nonzero when $i=j=k=l$ since $E(\varepsilon_i^4)=1$

OR when $E(\varepsilon_i^2 \varepsilon_k^2) = E(\varepsilon_i^2)\,E(\varepsilon_k^2) = 1$ (by indep) for which $i=j$ & $k=l$ but $i \neq k$ and there are $\binom{4}{2} = 6$ such combinations

$$\therefore \; E(|X|^4) = \sum_{i=1}^n a_i^4 + 6 \sum_{i<k}\sum a_i^2 a_k^2$$

$$\leq 3\left(\sum_{i=1}^n a_i^4 + 2\sum_{i<j} a_i^2 a_j^2\right)$$

$$\leq 3\left[\left(\sum_{i=1}^n a_i^2\right)^2\right]$$

$$\therefore \; E(|X|^4) \leq (\sqrt{3})^2 \left(\sum_{i=1}^n a_i^2\right)^2$$

$$\therefore \; \frac{1}{\{E(|X|^4)\}^{1/2}} \geq \frac{1}{\sqrt{3}} \frac{1}{\left(\sum_{i=1}^n a_i^2\right)}$$

$$\therefore \text{ from } (*) \Rightarrow \; E\left(\left|\sum a_i \varepsilon_i\right|\right) \geq \frac{1}{\sqrt{3}} \left\{\sum_{i=1}^n a_i^2\right\}^{1/2} \qquad \text{— II}$$

From I & II we get the result

#4] We need to show $M_r^{s-t} M_t^{r-s} \geq M_s^{r-t}$

where $M_r = E|X|^r$ $\quad r \geq s \geq t \geq 0$

ie we neet to show $(E|X|^r)^{s-t} [E(|X|^t)]^{r-s} \geq \{E(|X|^s)\}^{r-t}$

$$\Leftrightarrow \{E|X|^r\}^{\frac{s-t}{r-t}} \{E(|X|^t)\}^{\frac{r-s}{r-t}} \geq \{E(|X|^s)\}^1$$

$$\Leftrightarrow \{E(|X|^r)\}^{\frac{1}{\left(\frac{r-t}{s-t}\right)}} \{E(|X|^t)\}^{\frac{1}{\left(\frac{r-s}{s-t}\right)}} \geq E(|X|^s)$$

$$\Leftrightarrow \left\{\left\{E\left(|X|^{r\underbrace{\left(\frac{r-t}{s-t}\right)\left(\frac{s-t}{r-t}\right)}_{=1}}\right)\right\}^{\frac{1}{\left(\frac{r-t}{s-t}\right)}}\right\} \left\{E\left(|X|^{t\underbrace{\left(\frac{r-t}{r-s}\right)\left(\frac{r-s}{r-t}\right)}_{=1}}\right)\right\}^{\frac{1}{\left(\frac{r-t}{r-s}\right)}} \geq E(|X|^s)$$

$$\Leftrightarrow \left\{E\left(|X|^{r\left(\frac{s-t}{r-t}\right)}\right)^{\frac{1}{\left(\frac{r-t}{s-t}\right)}}\right\}^{\frac{1}{\left(\frac{r-t}{s-t}\right)}} \left\{E\left(|X|^{t\left(\frac{r-s}{r-t}\right)}\right)^{\left(\frac{r-t}{r-s}\right)}\right\}^{\frac{1}{\left(\frac{r-t}{s-t}\right)}} \geq E(|X|^s)$$

$$\left\{E\left(|X|^{\left(\frac{rs-rt}{r-t}\right)}\right)\right\}^{\frac{1}{\left(\frac{r-t}{s-t}\right)}} \left\{E\left(|X|^{\left(\frac{rt-st}{r-t}\right)}\right)\right\}^{\frac{1}{\left(\frac{r-t}{s-t}\right)}} \geq E(|X|^s)$$

Holders inequality $E|XY| \leq \{E(|X|^r)\}^{1/r} \{E(|Y|^s)\}^{1/s}$

where $r > 1$ & $\frac{1}{r} + \frac{1}{s} = 1$

Here

$$\left.\begin{aligned} r &= \frac{r-t}{s-t} \\ s &= \frac{r-t}{s-t} \\ X &= |X|^{\frac{rs-rt}{r-t}} \\ Y &= |X|^{\frac{rt-st}{r-t}} \end{aligned}\right\} \Rightarrow \begin{aligned} &\frac{1}{r} + \frac{1}{s} = \frac{s-t+r-s}{r-t} = 1 \checkmark \\ &X \cdot Y = |X|^{\frac{rs-rt+rt-st}{r-t}} = |X|^s \checkmark \end{aligned}$$

Now in our problem $r \geq s \geq t$

$$r-t \geq s-t \geq 0$$

$$\frac{r-t}{s-t} \geq 1$$

If $r = s \longrightarrow$ Liapunov's and we do not prove anything!

If $r > s \longrightarrow \frac{r-t}{s-t} > 1$

Apply Holder's ineq & the result follows.

The special case $m_4 m_1^2 \geq m_2^3$ follows if $r = 4$, $t = 1$, $s = 2$

5] Uses #4 & Liapunov's ineq.

Liapunov's inequality says $\{E|X|^r\}^{1/r}$ is $\uparrow$ in 'r' $\left\{\begin{array}{l}\text{for any}\\ r>0\end{array}\right.$

$\underbrace{\text{Radamacher}}$ random variables

Ques: $\varepsilon_1, \dots \varepsilon_n \sim$ iid random variables st $P(\varepsilon_i=1)=P(\varepsilon_i=-1)=\frac{1}{2}$

Let $a_i \in \mathbb{R}$

Show

$$A\left(\sum_{i=1}^{n} a_i^2\right)^{1/2} \leq E\left|\sum_{i=1}^{n} a_i\varepsilon_i\right| \leq B\left(\sum_{i=1}^{n} a_i^2\right)^{1/2}$$

Proof

Step 1 In Liapunov's ineq $E|X| \leq \{E|X|^2\}^{1/2}$

Take $X = \sum a_i \varepsilon_i$

$$\therefore \quad X^2 = \sum_{i=1}^{n} a_i^2 \varepsilon_i^2 + \sum_{i\neq j} a_i a_j \varepsilon_i \varepsilon_j$$

$$E(X^2) = \sum a_i^2 E(\varepsilon_i^2) + 0 \qquad \left\{\begin{array}{l}\because E(\varepsilon_i)=0\\ V(\varepsilon_i)=E(\varepsilon_i^2)=1\\ Cov(\varepsilon_i,\varepsilon_j)=0\end{array}\right.$$
$$= \sum a_i^2$$

$\therefore$ we get $E\left|\sum_{i=1}^{n} a_i\varepsilon_i\right| \leq \left\{\sum_{i=1}^{n} a_i^2\right\}^{1/2}$

Step 2

$$M_4 M_1^2 \geq M_2^3$$

$$E|X|^4 \{E|X|\}^2 \geq \{E|X|^2\}^3$$

$$E|X| \geq \frac{\{E|X|^2\}^{3/2}}{\{E|X|^4\}^{1/2}} = \frac{\{\sum a_i^2\}^{3/2}}{\{E|X|^4\}^{1/2}} \qquad \left\{ E(X^2) = \sum a_i^2 \right.$$

What we want is $\dfrac{1}{\{E|X|^4\}^{1/2}} \geqslant \dfrac{1}{D\{\Sigma a_i^2\}}$

then $E|X| \geqslant \dfrac{1}{D} \{\Sigma a_i^2\}^{1/2}$ $\left\{ \text{\& then call } 1/D \equiv A \right.$

To show $E(|X|^4) \leqslant \left[D\left(\sum_{i=1}^{n} a_i^2\right)\right]^2$

Now $E(|X|^4) = E(X^4) = E\left[\left(\sum_{i=1}^{n} a_i \varepsilon_i\right)^4\right]$

$$= E\left[\sum_i \sum_j \sum_k \sum_l a_i a_j a_k a_l \varepsilon_i \varepsilon_j \varepsilon_k \varepsilon_l\right]$$

$$= \sum_i \sum_j \sum_k \sum_l a_i a_j a_k a_l \underbrace{E(\varepsilon_i \varepsilon_j \varepsilon_k \varepsilon_l)}$$

is non-zero only when

① $i=j=k=l$

② $i=j$ & $k=l$ but $j<k$

$\binom{4}{2}$ pairs which are like this

$$= \sum_{i=1}^{n} a_i^4 + \binom{4}{2} \sum_{i=1}^{n} \sum_{\substack{k=1 \\ \boxed{i<k}}}^{n} a_i^2 a_k^2 \longrightarrow \text{check !!}$$

$$= \sum_{i=1}^{n} a_i^4 + 6 \sum_i \sum_{\substack{j \\ \boxed{i<j}}} a_i^2 a_j^2$$ change dummy variable

$$\leqslant 3\left(\sum_{i=1}^{n} a_i^4 + 2 \sum\sum_{i<j} a_i^2 a_j^2\right)$$

$$E(|X|^4) \leq 3\left(\sum_{i=1}^{n} a_i^2\right)^2$$

$$\therefore \quad E(|X|^4) \leq (\sqrt{3})^2 \left(\sum_{i=1}^{n} a_i^2\right)^2$$

$$\therefore \quad \{E(|X|^4)\}^{1/2} \leq \sqrt{3}\left[\sum_{i=1}^{n} a_i^2\right]$$

QED.

$$(a_1\varepsilon_1 + a_2\varepsilon_2 + a_3\varepsilon_3 + a_4\varepsilon_4)^2 = \sum_i (a_i\varepsilon_i)^2 + 2\sum_{i<j} a_i a_j \varepsilon_i \varepsilon_j$$

$$= \sum_i \sum_j a_i a_j \varepsilon_i \varepsilon_j$$

1 4 6 4 1 $(a+b)^4$

1 3 3 1 $(a+b)^3$

1 2 1 $(a+b)^2$

#1] If $X \geq 0$ and $\int X \, d\mu = 0 \Rightarrow \mu(X>0) = 0$

[Note if $X \ngeq 0$ & $\int X \, d\mu = 0 \nRightarrow X = 0$ always]

Proof By contradiction

$$p \longrightarrow q$$

We will show: $\mu\{\omega \in \Omega : X(\omega) > 0\} > 0 \Rightarrow \int X \cdot d\mu > 0$

(the $\sim q \rightarrow \sim p$)

Now $\{\omega \in \Omega : X(\omega) > 0\} = \bigcup_{n=1}^{\infty} \{\omega \in \Omega : X(\omega) > 1/n\}$

$$\mu\{\omega \in \Omega : X(\omega) > 0\} = \mu\left(\bigcup \{\omega \in \Omega ; X(\omega) > 1/n\}\right)$$

$$= \lim_{n \to \infty} \mu\left(\{\omega \in \Omega : X(\omega) > 1/n\}\right) > 0 \quad \because$$

[$A_n = \{X(\omega) > 1/n\}$ is $\uparrow$ seq]

$\Rightarrow \exists \, n_0$ st $\mu\{\omega \in \Omega : X(\omega) > 1/n_0\} > 0$ (*)

$\left\{\begin{array}{l} \lim a_n > 0 \\ \exists \, n_0 \text{ st } a_{n_0} > 0 \end{array}\right\}$

Now $\int X \cdot d\mu \geq \int_{\{\omega : X(\omega) \geq 1/n_0\}} X \cdot d\mu$

[$\int X \cdot d\mu \geq \int_A X \cdot d\mu$

$$\geq \int_{\{\omega : X(\omega) \geq 1/n_0\}} 1/n_0 \cdot d\mu$$

[since all $X(\omega) > 1/n_0$]

$\therefore \int X \cdot d\mu \geq \frac{1}{n_0} \mu\left(\{X(\omega) > 1/n_0\}\right)$

$\therefore \int X \cdot d\mu > 0$

{since $\mu\left(\{\omega : X(\omega) > 1/n_0\}\right) > 0$ from (*) & $\frac{1}{n_0} > 0$

#3] We have a measure space $(\Omega, \mathcal{A}, \mu)$

Let $\underbrace{\mu_0 \equiv \mu\Big|_{\mathcal{A}_0}}$ for a sub σ-field $\mathcal{A}_0 \subset \mathcal{A}$

[ie they agree on the same sets ie $\mu_0(A) = \mu(A) \quad \forall A \in \mathcal{A}_0$]

To show $\int X \cdot d\mu = \int X d\mu_0$ for any $\mathcal{A}_0$-measurable funct$^{\underline{n}}$ X

[Note If $\underline{X \text{ is } \mathcal{A}_0\text{-measurable}}$ then X is also $\mathcal{A}$-meas.

since $X^{-1}(\mathcal{B}) \subset \mathcal{A}_0 \subset \mathcal{A}$

$X: (\Omega, \mathcal{A}_0) \longrightarrow (R, \mathcal{B})$]

We will prove

Step 1 : Indicator funct$^{\underline{n}}$

Step 2 : Simple funct$^{\underline{n}}$

Step 3 : $X \geq 0$ and $X = \lim_n f_n$ where $f_n \geq 0$, simple funct$^{\underline{n}}$s & $f_n \uparrow X$

Step 4 : $X = X^+ - X^-$

Proof Step 1 $\int I_A \cdot d\mu \stackrel{?}{=} \int I_A d\mu_0$

since X is $\mathcal{A}_0$-meas $(A \in \mathcal{A}_0)$

$$\left.\begin{aligned} \int I_A \cdot d\mu &= \mu(A) \\ \int I_A d\mu_0 &= \mu_0(A) = \mu(A) \end{aligned}\right\} \Longrightarrow \int I_A d\mu = \int I_{A_0} d\mu$$

Step 2 From ① $\Rightarrow$ $c \int I_A \cdot d\mu = c \int I_A \cdot d\mu_0$ for any const c

$\therefore$ it holds for simple functions$(X) = \sum_{i=1}^{m} c_i I_{A_i}$

Step 3. To show $\int X \cdot d\mu = \int X \cdot d\mu_0$ for any $X \geq 0$

Now $X(\omega) = \lim_{n \to \infty} f_n(\omega)$ st $f_n \geq 0$, simple & $f_n \uparrow X$ (

$$\int X \cdot d\mu = \int \lim_{n\to\infty} f_n \, d\mu$$

$$= \lim_{n\to\infty} \int f_n \, d\mu \quad \left\{ \begin{array}{l} \text{By MCT for } X_n \uparrow X \ \& \ X_n \geq 0, X \geq 0 \\ \lim_{n\to\infty} \int X_n = \int \lim_{n\to\infty} X_n = \int X \end{array} \right.$$

$$= \lim_{n\to\infty} \int f_n \, d\mu_0 \quad \{\text{by step 2 for simple funct}^n\}$$

$$= \int \lim_{n\to\infty} f_n \cdot d\mu_0 \quad \{\text{by MCT}$$

$$= \int X \cdot d\mu_0 \quad \{\because X = \lim_{n\to\infty} f_n \quad ($$

Step 4. Any arbitrary funct^n $X = X^+ - X^-$ (can be written as)

Now
$$\int X^+ d\mu = \int X^+ d\mu_0$$
$$\int X^- d\mu = \int X^- d\mu_0$$
$\Bigg\{$ X^+ & X^- are both ≥ 0 by definition $\therefore$ using step 3

$$\int X^+ \cdot d\mu - \int X^- \cdot d\mu = \int X^+ d\mu_0 - \int X^- d\mu_0 \quad \left\{ \begin{array}{l} \text{holds unless} \\ \text{both } \int X^+, \int X^- = \pm\infty \end{array} \right.$$

[So we need at least one of $\int X^+ d\mu$ or $\int X^- d\mu < \infty$ (
which automatically implies one of $\int X^+ d\mu_0$ or $\int X^- d\mu_0 < \infty$
also since $\int X^+ d\mu = \int X^+ d\mu_0$ & $\int X^- d\mu = \int X^- d\mu_0$ always]

$|X| = X^+ + X^-$ ← Aside

2] Show

$$\int_A X \cdot d\mu \begin{cases} = 0 \\ \geq 0 \end{cases} \quad \forall \; A \in \mathcal{A} \implies X = \begin{cases} = 0 & ae \\ \geq 0 & ae. \end{cases}$$

$\boxed{\text{Proof}}$

Part 1 Since $\int_A X \cdot d\mu$ is defined, then X must be measurable on $\mathcal{A}$.

Step 1 Now $\{X > 0\} \in \mathcal{A}$ [X is measurable

$$\text{if} \int_{\{x>0\}} X \cdot d\mu = 0 \implies \int X \, I_{\{X>0\}} \cdot d\mu = 0$$

$$\implies \int X^+ d\mu = 0 \qquad \Big\{ X \cdot I_{\{X>0\}} = X^+$$

$$\implies X^+ = 0 \text{ a.e.} \qquad \begin{cases} \text{since } X^+ \geq 0 \text{ always} \\ \text{proved in \#1} \end{cases}$$

Step 2 Now $\{\omega : X(\omega) \leq 0\} \in \mathcal{A}$ $\begin{cases} \because X^{-1}([-\infty, 0]) \subset \mathcal{A} \text{ since } \mathcal{A} \text{ is meas} \\ \because X^{-1}(\mathcal{B}) \subset \mathcal{A} \end{cases}$

$$\therefore \text{if} \int_{\{X \leq 0\}} X \cdot d\mu = 0 \implies \int X \, I_{\{X \leq 0\}} \, d\mu = 0$$

$$\implies \int -(X^-) \cdot d\mu = 0 \qquad \left\{ \because X^- = \begin{cases} -X & \text{if } X \leq 0 \\ = 0 & \text{ow} \end{cases} \right.$$

$$\implies \int X^- \cdot d\mu = 0$$

$$\implies X^- = 0 \text{ a.e} \qquad \begin{cases} \because \text{since } X^- \geq 0 \text{ always} \\ \int X^- = 0 \implies X^- = 0 \end{cases}$$

$\therefore$ we have

$\therefore$ We have $X^+ = 0$ a.e

$X^- = 0$ a.e

Now $\mu(\{w \in \Omega : X(\omega) \neq 0\}) \leq \mu(X^+ \neq 0) + \mu(X^- \neq 0) \leq 0$

$\{X \neq 0\} \subseteq \{X^- \neq 0\} \cup \{X^+ \neq 0\}$

$\therefore \quad \mu(X \neq 0) = 0 \implies X = 0$ a.e

Part 2 $\quad \int_A X \cdot d\mu \geq 0 \implies X \geq 0$ a.e

To show $X \geq 0$ a.e we can show $X^- = 0$ ae

which gives us $X = X^+ - X^- = X^+ \geq 0$ a.e

Aside $\quad X \geq 0$ ae $\iff X^+ - X^- \geq 0$ ae

$$\Rightarrow \mu(X^+ - X^- < 0) = 0$$

$$\Rightarrow \mu\left(\left\{ X^+ I_{\{X>0\}} < -X I_{\{X<0\}} \right\}\right) = 0$$

$$\Rightarrow \mu\left(\left\{ w : X(w) < 0 \right\}\right) = 0$$ (the 2 sets are the same)

$\Rightarrow \quad X^- = 0$ a.e $\quad$ [since $X^- \geq 0$ always by definition

Now let $A \equiv \{ \omega : X(\omega) < 0 \}$

$$\int_A X \cdot d\mu = \int -X^- \cdot d\mu \geq 0 \qquad \text{(given)}$$

$$\Rightarrow \int X^- d\mu \leq 0$$

Now $X^- \geq 0$ always $\Rightarrow \int X^- d\mu \geq 0$.

$$\therefore \int X^- \cdot d\mu = 0$$

$$\Rightarrow X^- = 0 \quad \text{a.e} \quad \{ \text{by part 1}$$

$$\therefore X = X^+ \quad \text{a.e}$$

ie $X \geq 0$ a.e

4. Show $m_r^{s-t} \cdot m_t^{r-s} \geq m_s^{r-t}$, where $m_r = E|X|^r$, $r \geq s \geq t \geq 0$

Proof: Holder's Inequality $(E|X|^r)^{\frac{1}{r}} (E|Y|^s)^{\frac{1}{s}} \geq E|XY|$ where $\frac{1}{r}+\frac{1}{s}=1$, $r>1$

$m_r^{s-t} m_t^{r-s} \geq m_s^{r-t} \iff (E|X|^r)^{s-t} (E|X|^t)^{r-s} \geq (E|X|^s)^{r-t}$

$\iff (E|X|^r)^{\frac{s-t}{r-t}} (E|X|^t)^{\frac{r-s}{r-t}} \geq (E|X|^s)$

$\iff (E|X|^r)^{\frac{1}{\frac{r-t}{s-t}}} (E|X|^t)^{\frac{1}{\frac{r-t}{r-s}}} \geq E|X|^s$

$\iff \left(E|X|^{r \cdot \frac{s-t}{r-t} \cdot \frac{r-t}{s-t}}\right)^{\frac{1}{\frac{r-t}{s-t}}} \left(E|X|^{t \cdot \frac{r-s}{r-t} \cdot \frac{r-t}{r-s}}\right)^{\frac{1}{\frac{r-t}{r-s}}} \geq E|X|^s$

$\iff \left(E\left(|X|^{\frac{rs-rt}{r-t}}\right)^{\frac{r-t}{s-t}}\right)^{\frac{1}{\frac{r-t}{s-t}}} \left(E\left(|X|^{\frac{tr-ts}{r-t}}\right)^{\frac{r-t}{r-s}}\right)^{\frac{1}{\frac{r-t}{r-s}}} \geq E|X|^s$

Let $r' = \frac{r-t}{s-t}$, $s' = \frac{r-t}{r-s}$, $|X'| = |X|^{\frac{rs-rt}{r-t}}$, $|Y'| = |X|^{\frac{tr-ts}{r-t}}$

then $\frac{1}{r'} + \frac{1}{s'} = \frac{s-t+r-s}{r-t} = 1$, $|X'Y'| = |X|^{\frac{rs-rt+tr-ts}{r-t}} = |X|^s$

① If $r > s$ then $r' = \frac{r-t}{s-t} > 1$.

By Holder's Inequality: $(E|X'|^{r'})^{\frac{1}{r'}} (E|Y'|^{s'})^{\frac{1}{s'}} \geq E|X'Y'|$

$\iff \left(E\left(|X|^{\frac{rs-rt}{r-t}}\right)^{\frac{r-t}{s-t}}\right)^{\frac{1}{\frac{r-t}{s-t}}} \left(E\left(|X|^{\frac{tr-ts}{r-t}}\right)^{\frac{r-t}{r-s}}\right)^{\frac{1}{\frac{r-t}{r-s}}} \geq E|X|^s$

② If $r = s$ then $m_r^{s-t} \cdot m_t^{r-s} = m_s^{s-t} \cdot m_t^{s-s} = m_s^{s-t} = m_s^{r-t}$

From ① & ②, $m_r^{s-t} \cdot m_t^{r-s} \geq m_s^{r-t}$, $r \geq s \geq t \geq 0$

5. $\varepsilon_1, \cdots, \varepsilon_n$ iid r.v.'s s.t. $P(\varepsilon_i = 1) = P(\varepsilon_i = -1) = \frac{1}{2}$. $a_i \in \mathbb{R}$, $i = 1, 2, \cdots, n$.

show $A\left(\sum_{i=1}^n a_i^2\right)^{\frac{1}{2}} \le E\left|\sum_{i=1}^n a_i \varepsilon_i\right| \le B\left(\sum_{i=1}^n a_i^2\right)^{\frac{1}{2}}$ for some constants A and B

Proof: ① Let $X = \sum_{i=1}^n a_i \varepsilon_i \Rightarrow X^2 = (\sum a_i \varepsilon_i)^2 = \sum_{i=1}^n a_i^2 \varepsilon_i^2 + \sum\sum_{i \ne j} a_i a_j \varepsilon_i \varepsilon_j$

$E(\varepsilon_i^2) = 1$, $E(\varepsilon_i) = 0$, $cov(\varepsilon_i, \varepsilon_j) = 0 = E(\varepsilon_i \varepsilon_j) - E(\varepsilon_i) E(\varepsilon_j) = E(\varepsilon_i \varepsilon_j)$ for $i \ne j$

$\therefore EX^2 = \sum_{i=1}^n a_i^2 E(\varepsilon_i^2) + \sum\sum_{i \ne j} a_i a_j E(\varepsilon_i \varepsilon_j) = \sum_{i=1}^n a_i^2$

By Lyapunov's Inequality. $E|X| \le (E|X|^2)^{\frac{1}{2}}$

we have $E\left|\sum_{i=1}^n a_i \varepsilon_i\right| \le \left(\sum_{i=1}^n a_i^2\right)^{\frac{1}{2}}$

$\therefore E\left|\sum_{i=1}^n a_i \varepsilon_i\right| \le B\left(\sum_{i=1}^n a_i^2\right)^{\frac{1}{2}}$ with $B = 1$

② Let $X = \sum_{i=1}^n a_i \varepsilon_i$, from ① we have $E|X|^2 = \sum_{i=1}^n a_i^2$

In problem 4, we've showed $m_r^{s-t} m_t^{r-s} \ge m_s^{r-t}$, $r \ge s \ge t \ge 0$

letting $r = 4$, $s = 2$, $t = 1 \Rightarrow m_4 m_1^2 \ge m_2^3$

i.e. $(E|X|^4)(E|X|)^2 \ge (E|X|^2)^3$

$$\Leftrightarrow (E|X|)^2 \ge \frac{(E|X|^2)^3}{E|X|^4}$$

$$\Leftrightarrow E|X| \ge \frac{(E|X|^2)^{3/2}}{(E|X|^4)^{\frac{1}{2}}}$$

we want to show $E|X| \ge A\,(E|X|^2)^{\frac{1}{2}} = A\left(\sum_{i=1}^n a_i^2\right)^{\frac{1}{2}}$

If we have $\frac{1}{(E|X|^4)^{\frac{1}{2}}} \ge \frac{A}{E|X|^2}$, then $E|X| \ge \frac{(E|X|^2)^{3/2}}{(E|X|^4)^{\frac{1}{2}}} \ge \frac{A(E|X|^2)^{3/2}}{E|X|^2} = A(E|X|^2)^{\frac{1}{2}}$

So just show $(E|X|^4)^{\frac{1}{2}} \le \frac{E|X|^2}{A} \Leftrightarrow$ show $E|X|^4 \le \frac{(E|X|^2)^2}{A^2} = \frac{(\sum a_i^2)^2}{A^2}$

$E|X|^4 = EX^4 = E\left(\sum_{i=1}^n a_i \varepsilon_i\right)^4 = E\left|\sum_i\sum_j\sum_k\sum_l a_i a_j a_k a_l \varepsilon_i \varepsilon_j \varepsilon_k \varepsilon_l\right)$

$= \sum_i\sum_j\sum_k\sum_l a_i a_j a_k a_l E(\varepsilon_i \varepsilon_j \varepsilon_k \varepsilon_l)$

(i) $i = j = k = l \Rightarrow E(\varepsilon_i \varepsilon_j \varepsilon_k \varepsilon_l) = E(\varepsilon_i^4) = 1$

(ii) $i = j = p \ne q = k = l \Rightarrow E(\varepsilon_i \varepsilon_j \varepsilon_k \varepsilon_l) = E(\varepsilon_p^2 \varepsilon_q^2) = cov(\varepsilon_p^2, \varepsilon_q^2) + E(\varepsilon_p^2) E(\varepsilon_q^2) = 0 + 1 = 1$

(iii) All other cases $\Rightarrow E(\varepsilon_i \varepsilon_j \varepsilon_k \varepsilon_l) = cov(\varepsilon_i \varepsilon_j, \varepsilon_k \varepsilon_l) + E(\varepsilon_i \varepsilon_j) E(\varepsilon_k \varepsilon_l) = 0 + 0 = 0$

$\therefore E|X|^4 = \sum_{i=1}^n a_i^4 E(\varepsilon_i^4) + \binom{4}{2} \sum\sum_{i>j} a_i^2 a_j^2 E(\varepsilon_i^2 \varepsilon_j^2) = \sum_{i=1}^n a_i^4 + 6 \sum\sum_{i>j} a_i^2 a_j^2$

$\le 3\left(\sum_{i=1}^n a_i^4 + \sum\sum_{i \ne j} a_i^2 a_j^2\right) = 3\left(\sum\sum_{i=j} a_i^2 a_j^2 + \sum\sum_{i \ne j} a_i^2 a_j^2\right) = 3 \sum_i\sum_j a_i^2 a_j^2 = 3\left(\sum_{i=1}^n a_i^2\right)^2$

$\therefore (E|X|^4)^{\frac{1}{2}} \le \sqrt{3}\, E|X|^2 \quad \therefore E\left|\sum_{i=1}^n a_i \varepsilon_i\right| \ge A\left(\sum_{i=1}^n a_i^2\right)^{\frac{1}{2}}$ with $A = \frac{1}{\sqrt{3}}$