Homework 5 - Due Thursday, Dec. 8th

1. Exercise 3.5.3 page 54.

2. Exercise 3.5.6 page 57.
   (Hint: Use Ex. 3.5.3 above, if you need to).

3. If X is a non-negative random variable satisfying $\int_0^\infty \sqrt{P(X>t)}\, dt < \infty$, then we say that $X \in \mathcal{L}_{2,1}$. Show that <u>if</u> $X \in \mathcal{L}_{2,1}$ <u>then</u> $X \in \mathcal{L}_2$.

3) Show if $X \geq 0$ and $X \in \mathcal{L}_{2,1}$ then $X \in \mathcal{L}_2$

ie $\displaystyle\int_0^\infty \sqrt{P(X \geq t)}\, dt < \infty \rightarrow \int_0^\infty X^2 \, d\mu < \infty$

Case I X is bdd and $X \geq 0$

By Markov inequality $P(X \geq t) \leq \dfrac{E(X^2)}{t^2}$

$$\therefore \quad t^2 P(X \geq t) \leq E(X^2)$$

$$t \sqrt{P(X \geq t)} \leq \sqrt{E(X^2)}$$

Now $$E(X^r) = \int_0^\infty r t^{r-1} P(X > t)\, dt$$

$$\therefore E(X^2) = \int_0^\infty 2t\, P(X>t)\, dt$$

$$= 2 \int_0^\infty t\sqrt{P(X>t)}\, \sqrt{P(X>t)}\, dt$$

$$< 2 \int_0^\infty \sqrt{E(X^2)}\, \sqrt{P(X>t)}\, dt$$

$$= 2 \sqrt{E(X^2)} \int_0^\infty \sqrt{P(X>t)}\, dt$$

Now $E(X^2) = 0 \iff X = 0$ as and hence $E(X^2) < \infty$ & $X \in \mathcal{L}_2$.

$$\frac{E(X^2)}{\sqrt{E(X^2)}} < 2 \int_0^\infty \sqrt{P(X>t)}\, dt \quad \text{if } X > 0$$

$$\sqrt{E(X^2)} < 2 \int_0^\infty \sqrt{P(X>t)}\, dt$$

$$E(X^2) < 4 \left(\int_0^\infty \sqrt{P(X>t)}\, dt \right)^2 < \infty$$

$\therefore \quad X \in \mathcal{L}_2$

Case II   If X is not bdd.

Let $Y_n = X I_{[X \leq n]}$ and $X \in \mathcal{L}_{2,1}$

$Y_n \geq 0$ and $Y_n < \infty$ for a given n.

Also. $\lim\limits_{n\to\infty} Y_n = X$ ie $Y_n \uparrow X$

$\therefore \quad Y_n^2 \longrightarrow X^2$ $\{$ If $X_n \to X$ then $g(X_n) \to g(X)$ for $g \equiv$ continuous

Since Y_n are bdd by case I we have. $\sqrt{E(Y_n^2)} < 2 \int_0^\infty \sqrt{P(Y_n > t)}\, dt$

By MCT $Y_n^2 \to X^2$ and $Y_n \geq 0 \Rightarrow \int Y_n^2 \, d\mu \longrightarrow \int X^2 \, d\mu$

ie $\lim\limits_{n\to\infty} E(Y_n^2) = E(X^2)$

$\therefore \quad \lim\limits_{n\to\infty} \sqrt{E(Y_n^2)} = \sqrt{E(X^2)} < 2 \lim\limits_{n\to\infty} \int_0^\infty P(Y_n > t)\,dt$

$\sqrt{E(X^2)} < 2 \int_0^\infty P(X > t)\,dt$ $\{$ by MCT

$E(X^2) < 4\left(\int_0^\infty P(X > t)\,dt\right) < \infty$

$\therefore \quad X_2 \in \mathcal{L}_2$

2) a) $\xi \cong Unif(0,1)$ and $X_n \equiv \frac{n}{\log n} I_{[0,1/n]}(\xi)$ for $n \geq 3$

Show $\int X_n \, dP \longrightarrow 0$ and $\{X_n\}$ are u.i even though they are not dominate by any fixed r.v Y which is integrable

Sol

(i) Now $\int X_n \, dP \longrightarrow 0$ means $E(X_n) \longrightarrow 0$

Consider $E(X_n) = E\left(\frac{n}{\log n} I_{[0,1/n]}(\xi)\right) = \frac{n}{\log n} E\left(I_{[0,1/n]}(\xi)\right)$

$$= \frac{n}{\log n} P(\xi \in [0,1/n])$$

$$= \frac{n}{\log n}\left(\frac{1}{n}\right)$$

$$= \frac{1}{\log n}$$

$$\longrightarrow 0 \qquad \text{as } n \longrightarrow \infty$$

(ii) To show X_n's are u.i we will use Vitali's theorem for which we need $X_n \in \mathcal{L}_r$ & $X_n \xrightarrow{P} X$ (some X) and some $r > 0$

Now let $X = 0$ consider $P(|X_n - X| > \varepsilon) = P(X_n > \varepsilon)$

$$= P\left(\frac{n}{\log n} I_{[0,1/n]}(\xi) > \varepsilon\right)$$

$$= P(\xi \in [0,1/n]) = \frac{1}{n} \longrightarrow 0$$

$\because \frac{n}{\log n} I_{[0,1/n]}(\xi) = \begin{cases} \frac{n}{\log n} & \text{when } \xi \in [0,1/n] \\ 0 & \text{OW} \end{cases}$

$\therefore P(|X_n - X| > \varepsilon) \longrightarrow 0$ as $n \longrightarrow \infty$

ie $X_n \xrightarrow{P} 0$

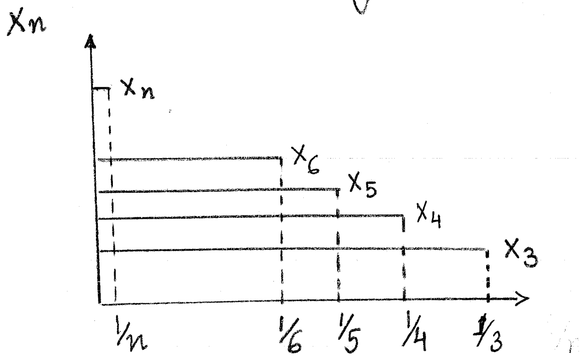
Also $E(|X_n|) = E(X_n) \longrightarrow 0$ as $n \to \infty$

$\therefore E(|X_n|) < \infty \implies X_n \in \mathcal{L}_1$

Now $E|X_n| \longrightarrow 0 = E|X|$ $\therefore$ by Vitali's theorem $\{X_n\}$ are u.i

iii) We must show that for any r.v st $X_n \leq Y$, Y is not integrable

We will show that the smallest possible Y st $X_n \leq Y$ is not integrable and $\therefore \nexists$ any Y st $X_n \leq Y$ and Y is integrable



Let $Y = \begin{cases} X_3 & \text{on } [1/4, 1/3] \\ X_4 & \text{on } [1/5, 1/4] \\ X_5 & \text{on } [1/6, 1/5] \\ \vdots & \end{cases}$

Define $$Y \equiv \sum_{n=3}^{\infty} \left(\frac{n}{\log n}\right) I_{[1/n+1,\ 1/n]}(\xi_g)$$

$$E|Y| = E\left[\sum_{n=3}^{\infty} \frac{n}{\log n} I_{[1/n+1,\ 1/n]}\right] = \sum_{n=3}^{\infty} \frac{n}{\log n} E\left(I_{[1/n+1,\ 1/n]}(\xi_g)\right)$$

$$= \sum_{n=3}^{\infty} \left(\frac{n}{\log n}\right)\left(\frac{1}{n(n+1)}\right) = \sum_{n=3}^{\infty} \frac{1}{(n+1)\log n}$$

$\longrightarrow \infty$ as $n \to \infty$ $\quad \left\{ \sum_{n=1}^{\infty} \frac{1}{n(\log n)^p} \to \infty \quad \forall p \geq 1 \right.$

$\therefore Y$ is not integrable.

(b) $Y_n = nI_{[0,1/n]}(\xi) - nI_{[1/n,2/n]}(\xi)$ Show $\int Y_n \longrightarrow 0$ but Y_n's are not u.i

$$E(Y_n) = E\left(nI_{[0,1/n]}(\xi) - nI_{[1/n,2/n]}(\xi)\right) = n\left(\frac{1}{n}\right) - n\left(\frac{1}{n}\right) = 0$$

$\therefore \int Y_n \, dP \longrightarrow 0$

To show Y_n's are not u.i we use Vitali's theorem for which we need $Y_n \in \mathcal{L}_1$ and $Y_n \xrightarrow{P} Y$ (some Y)

Let $Y=0$ consider $P(|Y_n - Y| > \varepsilon) = P(|Y_n| > \varepsilon)$

$$= P\left(\left|nI_{[0,1/n]}(\xi) - nI_{[1/n,2/n]}(\xi)\right| > \varepsilon\right)$$

Now $\left|nI_{[0,1/n]}(\xi) - nI_{[1/n,2/n]}(\xi)\right| = \begin{cases} n & \text{if } \xi \in [0, 2/n] \\ 0 & \text{ow} \end{cases}$

So $P(|Y_n - Y| > \varepsilon) = P(\xi \in [0, 2/n]) = \frac{2}{n} \longrightarrow 0$ as $n \to \infty$

$\therefore Y_n \xrightarrow{P} Y = 0$

Now $E(|Y_n|) = E\left|nI_{[0,1/n]}(\xi) - nI_{[1/n,2/n]}(\xi)\right| = E\left(\left|nI_{[0,1/n]}(\xi)\right|\right) + E\left(\left|nI_{[1/n,2/n]}(\xi)\right|\right)$

$= n\left(\frac{1}{n}\right) + n\left(\frac{1}{n}\right)$

$\left\{ \begin{array}{l} |I_A + I_B| = |I_A| + |I_B| \\ A \,\&\, B \text{ are disjoint.} \end{array} \right.$

$= 2 \not\to 0$

$\therefore E|Y_n| \not\longrightarrow E|Y|$

$\Rightarrow \{Y_n\}$ are not u.i

$X \geq 0$ with a df F. We have $E(X) = \int_0^\infty P(X \geq t)\,dt = \int_0^\infty [1 - F(t)]\,dt$

Show $\int_{[X \geq \lambda]} X\,dP = \lambda P(X \geq \lambda) + \int_\lambda^\infty P(X \geq y)\,dy$.

Now $\int_{[X \geq \lambda]} X\,dP = \int X I_{[X \geq \lambda]}\,dP = E(X I_{[X \geq \lambda]})$

$$= \int_0^\infty P(X I_{[X \geq \lambda]} \geq y)\,dy$$

$$= \int_0^\lambda P(X I_{[X \geq \lambda]} \geq y)\,dy + \int_\lambda^\infty P(X I_{[X \geq \lambda]} \geq y)\,dy$$

Now $\{\omega \in \Omega \mid X(\omega) I_{[X(\omega) \geq \lambda]} \geq y\} = \{\omega \in \Omega \mid X(\omega) \geq \lambda\}$ for $y \in [0, \lambda]$

$\because$ $X(\omega) I_{[X(\omega) \geq \lambda]} = \begin{cases} X(\omega) & \text{if } X(\omega) \geq \lambda \\ 0 & \text{ow} \end{cases}$

$\{\omega \in \Omega \mid X(\omega) I_{[X(\omega) \geq \lambda]} \geq y\} = \{\omega \in \Omega \mid X(\omega) \geq y\}$ for $y \in [\lambda, \infty]$

$X(\omega) I_{[X(\omega) \geq \lambda]} = \begin{cases} X(\omega) & \text{if } X(\omega) \geq \lambda \\ 0 & \text{ow} \end{cases}$

Since $y \geq \lambda > 0 \Rightarrow X(\omega)$ must be $\geq \lambda$ and also if we want $X(\omega) I_{[X(\omega) \geq \lambda]} \geq y$ then $X(\omega) \geq y$ which automatically implies $X(\omega) \geq \lambda$

$$\therefore \int_{[X \geq \lambda]} X\,dP = \int_0^\lambda P(X \geq \lambda)\,dy + \int_\lambda^\infty P(X \geq y)\,dy$$

$$= \left(\int_0^\lambda dy\right) P(X \geq \lambda) + \int_\lambda^\infty P(X \geq y)\,dy$$

$$= \lambda P(X \geq \lambda) + \int_\lambda^\infty P(X \geq y)\,dy$$

b) $\{X_n : n \geq 1\}$ is u.i – Show when $P(|X_n| \geq y) \leq P(Y \geq y)$ for $Y \in \mathcal{L}_1$ & $y >$

ie To show $\sup_n \{E\, |X_n|\, I_{[|X_n| \geq \lambda]}\} \longrightarrow 0$ as $\lambda \to \infty$

$$E\left(|X_n|\, I_{[|X_n| \geq \lambda]}\right) = \lambda P(|X_n| \geq \lambda) + \int_\lambda^\infty P(|X_n| \geq y)\, dy \qquad \{\text{from (a)}$$

$$\leq \lambda P(Y \geq \lambda) + \int_\lambda^\infty P(Y \geq y)\, dy$$

$$= E\left(Y I_{[|Y| \geq \lambda]}\right)$$

$$\leq E\left(|Y|\, I_{[|Y| \geq \lambda]}\right)$$

$$\sup_n E\left(|X_n|\, I_{[|X_n| \geq \lambda]}\right) \leq \sup_n E\left(|Y|\, I_{[|Y| \geq \lambda]}\right)$$

Now $P([|Y| \geq \lambda]) \leq \dfrac{E\,|Y|}{\lambda}$ Markov's Ineq

$$\frac{E|Y|}{\lambda} \longrightarrow 0 \text{ as } \lambda \to \infty \qquad \{\text{since } \underset{Y \in \mathcal{L}_1}{E|Y| = c < \infty}$$

$\therefore\ P(|Y| \geq \lambda) < \delta_\varepsilon \Rightarrow \int_{[|Y| \geq \lambda]} |Y|\, dP \longrightarrow 0$ $\{$ as $\lambda \to \infty$ by abs continuity of $\int$

$$\Rightarrow \sup_n E\left(|Y|\, I_{[|Y| \geq \lambda]}\right) \longrightarrow 0 \text{ as } \lambda \to$$

$$\therefore \sup_n E\left(|X_n|\, I_{[|X_n| \geq \lambda]}\right) \longrightarrow 0 \text{ as } \lambda \to \infty$$

$$\left.\begin{array}{l} X \geq 0 \\ \int_0^\infty \sqrt{P(X>t)}\,dt \end{array}\right\} \Rightarrow EX^2 < \infty$$

$$\boxed{EX^r = \int_0^\infty r t^{r-1} P(X>t)\,dt}$$

Case I: RV X is bounded

<u>let $X=0$</u>

$$EX^2 = \int_0^\infty 2t\, P(X>t)\,dt$$

* $P(X>t)=0$ since t takes values from $[0,\infty)$, X can never be strictly larger

$$EX^2 = \int_0^\infty 2\cdot t\cdot 0\,dt = 0 < \infty$$

<u>Let $X>0$</u>

$$P(X>t) < \frac{EX^2}{t^2} \quad \text{(Markov)} \quad * X, t \text{ always positive}$$

$$\therefore EX^2 > t^2 \cdot P(X>t)$$

$$\therefore \sqrt{EX^2} > t\cdot\sqrt{P(X>t)}$$

$$EX^2 = \int_0^\infty 2t\,P(X>t)\,dt = 2\int_0^\infty t\cdot\sqrt{P(X>t)}\cdot\sqrt{P(X>t)}\,dt < 2\int_0^\infty \sqrt{EX^2}\cdot\sqrt{P(X>t)}\,dt$$

$$EX^2 < 2\cdot\sqrt{EX^2}\underbrace{\int_0^\infty \sqrt{P(X>t)}\,dt}_{\text{finite "A"}} = 2\cdot\sqrt{EX^2}\cdot A$$

$$\frac{EX^2}{\sqrt{EX^2}} < 2\cdot A$$

$$\therefore \sqrt{EX^2} < 2\cdot A$$

$$EX^2 < 4A^2 < \infty$$

Case 2: RV X not bounded

From case 1 we know that <u>if</u> X were bounded by some k, then $EX^2 < \infty$, we can say that some subsequence of X that is bounded by some k exists

$X1_{[X \le k]}$ denoted by X_k

and that $X_k \geq 0$

Note that $X1_{[X \le k]} \to X$ as $k \to \infty$

X_k's increasing.

$$\underbrace{X^2 1_{[X \le k]}}_{X_k^2} \to X^2$$

By MCT:

$$\lim_{k \to \infty} \int X_k^2 \, dp = \int X^2 \, dp$$

$$\lim_{k \to \infty} EX_k^2 = EX^2$$

$$EX^2 = \lim_{k \to \infty} EX_k^2 < \infty$$

Ex 5.6 $\quad \xi \cong U[0,1]$

$$X_n = \frac{n}{\log n} 1_{[0,\frac{1}{n}]}(\xi)$$

sufficient: If $\exists\, Y \in L_1$ st $|X_n| \leq Y \Rightarrow \{X_n\}$ is u.i

But not necessary cond^n

To show (i) $\int X_n \, dP \longrightarrow 0$

(ii) X_n are u.i.

(iii) X_n are not dominated by any fixed integrable r.v. Y

Part(i):- $\quad \int X_n \, dP \longrightarrow 0 \iff E(X_n) \longrightarrow 0$

Let $r=1, \quad X=0$

$$E(X_n) = E\left(\frac{n}{\log n} 1_{[0,\frac{1}{n}]}(\xi)\right) = \frac{n}{\log n} E\left(1_{[0,\frac{1}{n}]}(\xi)\right)$$

Taking Lim $n\to\infty$ on both Sides.

$$\lim_{n\to\infty} E(X_n) = \lim_{n\to\infty} \frac{n}{\log n} \times \frac{1}{n} = \lim_{n\to\infty} \frac{1}{\log n} = 0$$

$$\therefore \lim_{n\to\infty} E(X_n) = 0$$

$$E(X_n) \xrightarrow[n\to\infty]{} 0$$

Now we will show that

$$X_n \xrightarrow{P} X.$$

$$P(|X_n - X| > \varepsilon) = P(|X_n - 0| > \varepsilon) = P\left(\left|\frac{n}{\log n} 1_{[0,\frac{1}{n}]}(\xi)\right| > \varepsilon\right)$$

Now $X_n = \begin{cases} \frac{n}{\log n}, & \text{if } \xi \in [0, \frac{1}{n}] \\ 0, & \text{otherwise.} \end{cases}$

$\therefore \forall \varepsilon > 0,$

$|X_n| \geq \varepsilon$ is same as $X_n = \frac{n}{\log n} > 0$

$\therefore P(|X_n| > \varepsilon) = P\left(X_n = \frac{n}{\log n}\right) = P\left(\xi \in [0, \tfrac{1}{n}]\right)$

$= \frac{1}{n}.$

Taking $\lim n \to \infty$ on both sides.

$$\lim_{n \to \infty} P(|X_n| > \varepsilon) = \lim_{n \to \infty} \frac{1}{n} = 0.$$

Hence we have shown that

$$X_n \xrightarrow{P} X$$

Part (ii) We have to show X_n are u.i.

For this we take $r=1$, $X=0$.

Now, if we can show that $E|X_n| \longrightarrow E|X|$, then by Vitali's

it is equivalent to showing

$\{|X_n| : n \geq 1\}$ are u.i. r.v.s

$$E|X_n| = E(X_n) = E\left(\frac{n}{\log n} 1_{[0,\frac{1}{n}]}(\xi)\right)$$

$$E|X_n| = \frac{n}{\log n} E\left(1_{[0,\frac{1}{n}]}(\xi)\right) = \frac{n}{\log n} P\left(\xi \in [0,\tfrac{1}{n}]\right)$$

Taking Lim $n \to \infty$ on both sides.

$$\lim_{n\to\infty} E|X_n| = \lim_{n\to\infty} \frac{n}{\log n} * \frac{1}{n} = 0.$$

and $E(X| = E|0| = 0.$

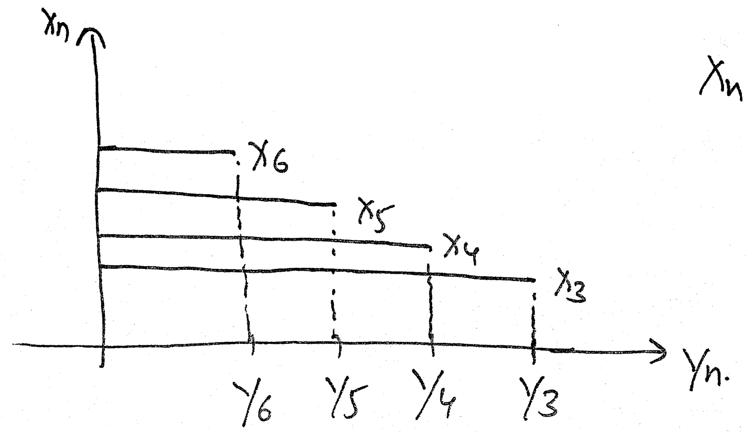
$\therefore \; E|X_n| \longrightarrow E|X|.$

Hence, $\{|X_n|, n \geq 1\}$ are u.i.

Part (iii) We have to come up with a r.v. Y, s.t $Y \geq X_n$ but Y is not integrable

We will show that smallest Y, for which $Y \geq X_n$ is not integrable

Hence, $\nexists Y$, s.t. $Y \geq X_n$ and Y is integrable.



$$X_n = \frac{n}{\log n} 1_{[0,\frac{1}{n}]}(\xi), \; n \geq 3.$$

The smallest $Y \geq X_n$ is.

$Y = X_3$, in interval $\left[\frac{1}{4}, \frac{1}{3}\right]$

$Y = X_4$, in interval $\left[\frac{1}{5}, \frac{1}{4}\right]$

$Y = X_5$, in " $\left[\frac{1}{6}, \frac{1}{5}\right]$

$\vdots$

$Y = X_n$, " " $\left[\frac{1}{n+1}, \frac{1}{n}\right]$

$$\therefore \; Y = \sum_{n=3}^{\infty} \frac{n}{\log n} \mathbb{1}_{\left[\frac{1}{n+1}, \frac{1}{n}\right]}$$

$$E|Y| = E\left[\left|\sum_{n=3}^{\infty} \frac{n}{\log n} \mathbb{1}_{\left[\frac{1}{n+1}, \frac{1}{n}\right]}\right|\right]$$

$$= \sum_{k=3}^{\infty} \frac{n}{\log n} E\left[\mathbb{1}_{\left[\frac{1}{n+1}, \frac{1}{n}\right]}\right] = \sum_{k=3}^{\infty} \frac{\cancel{n}}{\log n} \cdot \frac{1}{\cancel{n}(n+1)}$$

$$= \sum_{k=3}^{\infty} \frac{1}{(n+1)\log n}.$$

Rudin

$$\sum_{n=1}^{\infty} \frac{1}{n(\log n)^p} \longrightarrow \infty \quad \text{if } p \geq 1$$

$E|Y| \longrightarrow \infty$ using

Theorm:- If $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$, Then Series $\sum_{n=1}^{\infty} a_n$ Converges if and only if the series,

$$\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots, \text{ Converges}$$

Let $a_n = \frac{1}{(n+1)\log(n)}$, $[(n+1)\log(n)] \uparrow$ $\therefore$, $a_n \searrow 0$

Then it can be shown $\sum_{k=0}^{\infty} 2^k a_{2^k} \longrightarrow \infty$

(b) We have to show

(i) $\int Y_n \, dP \longrightarrow 0$

(ii) Y_n are not u.i.

Part (i)

$$\int Y_n \, dP = E(Y_n) = E\left(n \, 1_{[0,\frac{1}{n}]}(\xi_e) - n \, 1_{[\frac{1}{n},\frac{2}{n}]}(\xi_e)\right)$$

$$= n\left[E\left(1_{[0,\frac{1}{n}]}(\xi_e)\right) - E\left(1_{[\frac{1}{n},\frac{2}{n}]}(\xi_e)\right)\right]$$

$$E(Y_n) = n\left[\frac{1}{n} - \frac{1}{n}\right] = 0.$$

Now, we show that $Y_n \xrightarrow{P} 0$, $\xi = 1$.

$$P(|Y_n - 0| > \varepsilon) = P(|Y_n| > \varepsilon), \quad \varepsilon > 0$$

$$= P\left(\left|n \, 1_{[0,\frac{1}{n}]}(\xi_e) - n \, 1_{[\frac{1}{n},\frac{2}{n}]}(\xi_e)\right| > \varepsilon\right)$$

$$\left|n \, 1_{[0,\frac{1}{n}]}(\xi_e) - n \, 1_{[\frac{1}{n},\frac{2}{n}]}(\xi_e)\right| = \begin{cases} n, & \xi_e \in [0,\frac{2}{n}] \\ 0, & \text{otherwise.} \end{cases}$$

$$\therefore P\left(\left|n \, 1_{[0,\frac{1}{n}]}(\xi_e) - n \, 1_{[\frac{1}{n},\frac{2}{n}]}(\xi_e)\right| > \varepsilon\right) = P(|Y_n| = n)$$

$$= P\left(\xi_e \in [0,\tfrac{2}{n}]\right)$$

$$P(|Y_n| = n) = \frac{2}{n}$$

Taking limit on both sides

$$\lim_{n\to\infty} P(|Y_n|>\varepsilon) = \lim_{n\to\infty} P(|Y_n|=n) = \lim_{n\to\infty} \frac{2}{n} = 0.$$

Hence $Y_n \xrightarrow{P} 0.$

Now, we have to show that Y_n, $n\geq 1$, are not u.i.

It will be sufficient to show that

$$E|Y_n| \not\to E|Y|=0. \quad , \text{ using Vitali's theorem}$$

$$E|Y_n| = E\left[\left|n\, 1_{[0,\frac{1}{n}]}(\varepsilon) - n\, 1_{[\frac{1}{n},\frac{2}{n}]}(\varepsilon)\right|\right]$$

$$= n\, E\left[\left| 1_{[0,\frac{1}{n}]}(\varepsilon) - n\, 1_{[\frac{1}{n},\frac{2}{n}]}(\varepsilon)\right|\right]$$

$$= n\left[E\left(\left|1_{[0,\frac{1}{n}]}(\varepsilon)\right|\right) + E\left(\left|1_{[\frac{1}{n},\frac{2}{n}]}(\varepsilon)\right|\right)\right]$$

$$= n\left[\frac{1}{n}+\frac{1}{n}\right] = 2 \neq 0$$

Hence, $E|Y_n| \not\to 0$

$\therefore$ Y_n are not u.i.

② $\int_\lambda^\infty P(X \cdot 1_{[X \geq \lambda]} \geq y)\, dy$

$y \in [\lambda, \infty)$

$$\{w : X(w) \cdot 1_{[X\geq\lambda]}(w) \geq y\} = \{w : X(w) \geq y\}$$

$$\therefore \int_\lambda^\infty P(X \cdot 1_{[X\geq\lambda]} \geq y)\, dy = \int_\lambda^\infty P(X(w) \geq y)\, dy$$

$$= \int_\lambda^\infty P(X \geq y)\, dy$$

#

b) to show $\{X_n : n \geq 1\}$ is u.i

$\Leftrightarrow \sup_n E\, |X_n| \cdot 1_{[|X_n| \geq \lambda]} \longrightarrow 0 \quad \lambda \to \infty$

$$\sup_n E\, |X_n| \cdot 1_{[|X_n|\geq\lambda]} = \sup_n \{\lambda P(|X_n| \geq \lambda) + \int_\lambda^\infty P(|X_n| \geq y)\, dy\}$$

$$\leq \sup_n \{\lambda P(Y \geq \lambda) + \int_\lambda^\infty P(Y \geq y)\, dy\}$$

$$= \sup_n E\, Y \cdot 1_{[Y\geq\lambda]} = E\, Y \cdot 1_{[Y \geq \lambda]}$$

$$\leq E\, |Y| \cdot 1_{[|Y|\geq\lambda]}$$

now since $Y \in \mathcal{L}_1$

$\therefore$ as $\lambda \to \infty \quad E|Y| 1_{[|Y|\geq\lambda]} \to 0$

To ~~prove~~ prove this let $A = [|Y| \geq \lambda]$

$P(A) \leq \frac{E|Y|}{\lambda} = \frac{\text{constant}}{\lambda}$ (for $Y \in \mathcal{L}_1$) $\therefore P(A) \to 0$ as $\lambda \to \infty$

$\therefore$ using Theorem 2.5 (Absolute continuity of the Integral)

we have $E\, |Y|\, 1_{[|Y| \geq \lambda]} \to 0$ as $\lambda \to \infty$

$\therefore$ for $\sup_n E|X_n| \cdot 1_{[|X_n|\geq\lambda]} \leq E|Y| \cdot 1_{[|Y|\geq\lambda]}$

$\therefore \sup_n E|X_n| \cdot 1_{[|X_n|\geq\lambda]} \to 0$ as $\lambda \to \infty$ $\therefore \{X_n : n \geq 1\}$ is u.i

HW P54 3.5.4

Fact: $X \geq 0$ with df F

then we have $EX = \int_0^\infty P(X \geq y)\,dy = \int_0^\infty (1 - F(y))\,dy$

To show:

a) $X \geq 0$ $\lambda \geq 0$

$$\int_{[X \geq \lambda]} X\,dP = \lambda P(X \geq \lambda) + \int_\lambda^\infty P(X \geq y)\,dy$$

Proof: $\int_{[X \geq \lambda]} X\,dP = \int X \cdot 1_{[X \geq \lambda]}\,dP = E X \cdot 1_{[X \geq \lambda]}$

for $X \geq 0$ $\therefore$ $X \cdot 1_{[X \geq \lambda]} \geq 0$

using Fact, we have

$$E X \cdot 1_{[X \geq \lambda]} = \int_0^\infty P(X \cdot 1_{[X \geq \lambda]} \geq y)\,dy.$$

$$= \int_0^\lambda P(X \cdot 1_{[X \geq \lambda]} \geq y)\,dy + \int_\lambda^\infty P(X \cdot 1_{[X \geq \lambda]} \geq y)\,dy$$

① $y \in [0, \lambda)$

$\{w: X(w) \cdot 1_{[X \geq \lambda]}(w) \geq y\} = \{w: X(w) \geq \lambda\}$

$\therefore$ $P(X \cdot 1_{[X \geq \lambda]} \geq y) = P(X(w) \geq \lambda)$

$\therefore$ $\int_0^\lambda P(X \cdot 1_{[X \geq \lambda]} \geq y)\,dy = \int_0^\lambda \underbrace{P(X(w) \geq \lambda)}_{\text{without } y \text{ . a constant}}\,dy$

$= P(X(w) \geq \lambda) \cdot \lambda$

HOMEWORK # 5

1) P is a prob meas. $X \geq 0$ with df F

a) $E(X) = \int_0^\infty P(X \geq y)\,dy = \int_0^\infty [1-F(y)]\,dy$

b) $X \geq 0$ and $\lambda \geq 0$

$$\int_{[X \geq \lambda]} X \cdot dP = \lambda P(X \geq \lambda) + \int_\lambda^\infty P(X \geq y)\,dy$$

c) Suppose $\exists\ Y \in \mathcal{L}_1$ st $P(|X_n| \geq y) \leq P(Y \geq y)$ $\forall\ y > 0$ & $n \geq 1$
then $\{X_n\}_{n \geq 1}$ is u.i

Proof b) Consider $\int_{[X \geq \lambda]} X \cdot dP = \int X\, I_{[X \geq \lambda]}\, dP = E\left(X\, I_{[X \geq \lambda]}\right)$

$$= \int_0^\infty P\left(X\, I_{[X \geq \lambda]} \geq y\right) dy \quad , y > 0 \qquad \{\text{by a}\}$$

$$= \int_0^\lambda P\left(X\, I_{[X \geq \lambda]} \geq y\right) dy + \int_\lambda^\infty P\left(X\, I_{[X \geq \lambda]} \geq y\right) dy$$

Now when $y \in [0, \lambda]$ Let

$X(\omega)\, I_{[X(\omega) \geq \lambda]} = \begin{cases} X(\omega) & \text{if } X(\omega) \geq \lambda \\ 0 & \text{ow} \end{cases}$

$A_1 = \{\omega \in \Omega \mid X(\omega)\, I_{[X(\omega) \geq \lambda]} \geq y\}$

$A_2 = \{\omega \in \Omega \mid X(\omega) \geq y\}$

$A_3 = \{\omega \in \Omega \mid X(\omega) \geq \lambda\}$

Now when $y \in [0, \lambda]$ $\quad \omega \in A_1 \Leftrightarrow X(\omega)\, I_{[X(\omega) \geq \lambda]} \geq y$

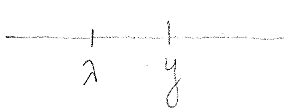
$\Leftrightarrow X(\omega) \geq \lambda$

$\Leftrightarrow \omega \in A_3$

When $y \in [\lambda, \infty)$ $\quad \omega \in A_1 \Leftrightarrow X(\omega)\, I_{[X(\omega) \geq \lambda]} \geq y$

$\Leftrightarrow X(\omega) \geq y$

$\Leftrightarrow \omega \in A_2$



$$\therefore \int_{[X \geq \lambda]} X \cdot dP = \int_0^\lambda P(X(\omega) \geq \lambda)\, dy + \int_\lambda^\infty P(X \geq y)\, dy$$

$$= P(X \geq \lambda) \int_0^\lambda dy + \int_\lambda^\infty P(X \geq y)\, dy$$

$$= \lambda P(X \geq \lambda) + \int_\lambda^\infty P(X \geq y)\, dy$$

c) To show $\sup_n E\left(|X_n| I_{[|X_n| \geq \lambda]}\right) \xrightarrow[\lambda \to \infty]{} 0$

Now $E\left(|X_n| I_{[|X_n| \geq \lambda]}\right) = \int_{[|X_n| \geq \lambda]} |X_n| dP$

$$= \int_{[|X_n| \geq \lambda]} |X_n| \cdot dP$$

$$= \int_\lambda^\infty P(|X_n| \geq y)\, dy + \lambda P(|X_n| \geq \lambda) \quad \text{from (b)}$$

$$\leq \int_\lambda^\infty P(Y \geq y)\, dy + \lambda P(Y \geq \lambda)$$

$$= \int_{[Y \geq \lambda]} Y \cdot dP = E\left(Y I_{[Y \geq \lambda]}\right) \leq E\left(|Y| I_{[|Y| \geq \lambda]}\right)$$

Now $E|Y| I_{[|Y| \geq \lambda]} \longrightarrow 0$ as $\lambda \to \infty$

by absolute continuity of integr $\{\because Y \in L_1\}$

2] a) $\xi \cong Unif(0,1)$ $\quad X_n = \frac{n}{\log n} I_{[0,1/n]}(\xi) \quad \forall\ n \geq 3$

Show X_n are u.i and $\int X_n\, dP \to 0$ even though X_n are not dominated by any fixed integrable Y

b) $Y_n = n I_{[0,1/n]}(\xi) - n I_{[1/n,\,2/n]}(\xi)$. Show Y_n's are not u.i but $\int Y_n\, dP \to 0$

Proof

a) To show $\int X_n\, dP \to 0$ ie $E(X_n) \to 0$

$$E(X_n) = E\left[\frac{n}{\log n} I_{[0,1/n]}(\xi)\right] = \frac{n}{\log n}\left[E\left(I_{[0,1/n]}(\xi)\right)\right]$$

$$= \frac{n}{\log n}\left[P(\xi \in [0,1/n])\right]$$

$$= \frac{n}{\log n} \times \frac{1}{n} = \frac{1}{\log n} \xrightarrow[n\to\infty]{} 0$$

To show u.i we use Vitali's theorem for which we need $X_n \in \mathcal{L}_1$ and $X_n \xrightarrow{P} X$ [We use $r=1$ and $X=0$]

To show $X_n \xrightarrow{P} X$ consider $P(|X_n - X| > \varepsilon) = P\left(\left|\frac{n}{\log n} I_{[0,1/n]}(\xi)\right| > \varepsilon\right)$

$$\leq P(|\xi \in [0,1/n]|)$$

$$= \frac{1}{n} \longrightarrow 0 \quad \text{as } n \to \infty$$

$\therefore X_n \xrightarrow{P} 0$

So by Vitali's theorem since $X_n \xrightarrow{P} 0$ and $E(X_n) \to 0 < \infty$ [ie $X_n \in \mathcal{L}_1$]

we have $E|X_n| \to E|X| \Rightarrow \{X_n\}$ are u.i

Let $y = \sum_{k=3}^{\infty} \frac{k}{\log k} I_{[\frac{1}{k+1}, \frac{1}{k}]}(\xi)$

$y \geq X_n$

$$E(y) = \sum_{k=3}^{\infty} \frac{k}{\log k} E\left(I_{[1/k+1, 1/k]}(\xi)\right)$$

$$= \sum_{k=3}^{\infty} \frac{k}{\log k} \frac{1}{k(k+1)} = \sum_{k=3}^{\infty} \left(\frac{1}{\log k}\right)\left(\frac{1}{k+1}\right)$$

$$= \infty$$

$\left\{ \text{if } \sum a_n \to a \ \& \ \sum b_n \to b \right.$
$\quad \sum a_n b_n \to ab$

b) $y_n = n I_{[0, 1/n]}(\xi) - n I_{[1/n, 2/n]}(\xi)$

$$E(y_n) = n P(\xi \in [0, 1/n]) - n P\left(\xi \in [1/n, 2/n]\right) = n\left(\frac{1}{n}\right) - n\left(\frac{1}{n}\right) = 0$$

ie $\int y_n \, dP \longrightarrow 0$

To show $\{y_n\}$ are not u.i we need to show any one cond^n of Vitali's theorem is violated $[A \Leftrightarrow B$ means $A^c \Leftrightarrow B^c]$

We first show $y_n \xrightarrow{P} 0$

Consider $P(|y_n| > \varepsilon) = P\left(\left|n I_{[0, 1/n]}(\xi) - n I_{[1/n, 2/n]}\right| > \varepsilon\right)$

Now $\xi(\omega) \in \begin{cases} [0, 1/n] \\ \text{OR} \\ [1/n, 2/n] \\ \text{OR} \\ [2/n, 1] \end{cases} \Rightarrow y_n = \begin{cases} 0 & \text{if } \xi \in [2/n, 1] \\ +n & \text{if } \xi \in [0, 1/n] \\ -n & \text{if } \xi \in [1/n, 2/n] \end{cases}$

$\therefore P(|y_n| > \varepsilon) \leq P\left(\{\omega \in \Omega \text{ st } \xi(\omega) \in [0, 1/n] \text{ OR } \xi(\omega) \in [1/n, 2/n]\}\right)$

$\leq P \ \xi \in (0, 2/n)$

$= \frac{2}{n} \longrightarrow 0$ as $n \to \infty$

$\therefore\ E(Y_n) = 0 \Rightarrow Y_n \in \mathcal{L}_1$ and we also have $Y_n \xrightarrow{P} 0$

We now show $E|Y_n| \not\to E|Y|$ where $Y=0 \quad \forall\ \omega \in \Omega$

$$E|Y_n| = E\left(\left| nI_{[0,1/n]}(\omega) - nI_{(1/n,2/n]}(\omega)\right|\right)$$

$$= E\left(\left|nI_{[0,1/n]}(\omega)\right| + \left|nI_{(1/n,2/n]}(\omega)\right|\right)$$

$\left\{ |I_A - I_B| = |I_A| + |I_B| \right.$ since supports are disjoint ie A & B are disjoint

$$= E\left(\left|nI_{[0,1/n]}(\omega)\right|\right) + E\left(\left|nI_{(1/n,2/n]}(\omega)\right|\right)$$

$$= n\left(\frac{1}{n}\right) + n\left(\frac{1}{n}\right)$$

$$= 2 \not\to 0$$

$$E|Y| = 0$$

So $E|Y_n| \not\to E|Y| \Rightarrow \{Y_n\}$ are not u.i

$X \geq 0$ st $\displaystyle\int_0^\infty \sqrt{P(X \geq t)}\, dt < \infty$ and we say $X \in \mathcal{L}_{2,1}$

without $\sqrt{\ }$ we have $\int_0^\infty P(X \geq t)\,dt = E(X)$

Show $X \in \mathcal{L}_{2,1} \Rightarrow X \in \mathcal{L}_2$ ie $E(X^2) < \infty$

Proof: Case I $\quad X = 0 \Rightarrow E(X^2) = 0 \Rightarrow X \in \mathcal{L}_2$

Case II $\quad X > 0$ and X is bdd

Now $E(X^r) = \int r t^{r-1} P(X > t)\, dt$

$$P(X > t) < \frac{E(X^2)}{t^2} \quad \text{Markov ineq}$$

$$E(X^2) > t^2 P(X > t)$$

$$\sqrt{E(X^2)} > t\sqrt{P(X > t)}$$

Now $E(X^2) = \displaystyle\int_0^\infty 2t\, P(X > t)\, dt$

$$= 2 \int_0^\infty \underbrace{t\sqrt{P(X>t)}}\ \sqrt{P(X>t)}\ dt$$

$$< 2 \int_0^\infty \sqrt{E(X^2)}\ \sqrt{P(X>t)}\ dt$$

$$\frac{E(X^2)}{\sqrt{E(X^2)}} < 2 \int_0^\infty \sqrt{P(X>t)}\, dt$$

$$\sqrt{E(X^2)} < 2 \int_0^\infty \sqrt{P(X>t)}\, dt$$

$$E(X^2) < 4 \left(\int_0^\infty \sqrt{P(X>t)}\ dt \right)^2 < \infty$$

Case III $\quad X$ is unbdd.

Let $\quad X_k = X\, I_{[X \le k]}$

$X_k \uparrow X \quad$ as $k \to \infty$

$X_k \ge 0 \quad \{$ since $X \ge 0$

$\therefore \quad X_k^2 \uparrow X^2$

$$\int X_k^2 \xrightarrow[k\to\infty]{} \int X^2 \qquad (\text{by MCT})$$

$$E(X^2) = \lim_{k\to\infty} \int X_k^2\, dP$$

Now $\quad X_k$'s are bdd and $\sqrt{E(X_k^2)} < 2 \int_0^\infty \sqrt{P(X_k > t)}\, dt$

$\Downarrow \qquad \{$by MCT

$$\sqrt{E(X^2)} \le 2 \int_0^\infty \sqrt{P(X>t)}\, dt$$

$$\therefore \quad E(X^2) \le 4 \left(\int_0^\infty \sqrt{P(X>t)}\ dt \right)^2$$

#3] $X \geq 0$ st $\int_0^\infty \sqrt{P(X>t)}\, dt < \infty \Rightarrow X \in \mathcal{L}_{2,1}$.

HINTS.

Case I X is bdd

Case II X is unbdd & we can write X as a lim of bdd funct^n

$$X I_{[|X| \leq M]} \uparrow X \quad \text{as } M \to \infty$$

& Then apply M.C.T.

Proof X is bdd

If $X = 0 \implies E(X^2) = 0$ & $X \in \mathcal{L}_2$ (no work needed)

To show $\int X^2 dP < \infty$ by using $\underbrace{\int X^2 dp \;----\leq}_{\text{show}} \int \sqrt{P(X>t)} < \infty$

Now by the markov ineq $P(X \geq t) \leq \dfrac{E(X^2)}{t^2}$

$$\therefore \quad t\sqrt{P(X \geq t)} \leq \sqrt{E(X^2)}$$

$$E(X^r) = \int_0^\infty r x^{r-1} P(X > x)\, dx \qquad [\text{Result}]$$

$$P(X > t) \leq \frac{E(X^2)}{t^2} \implies \sqrt{E(X^2)} \geq \sqrt{t^2 P(X > t)}$$

$$E(X^2) = \int_0^\infty 2x\, P(X > x)\, dx = \int_0^\infty 2t\, P(X > t)\, dt$$

$$\leq \int_0^\infty 2t \frac{E(X^2)}{t^2}\, dt$$

$$= E(X^2) \int_0^\infty 2\left(\frac{1}{t}\, dt\right)$$

3.5.3 (a) $X \geq 0$ with df F

then $E(X) = \int_0^\infty P(X \geq y)\,dy = \int_0^\infty [1 - F(y)]\,dy$ (easy to prove)

Show (b) If $X \geq 0$ & $\lambda \geq 0$

$$\int_{[X \geq \lambda]} X \cdot dP = \int X\, I_{[X \geq \lambda]}\, dP = E\left[X I_{[X \geq \lambda]} \right]$$

$$= \int_0^\infty P(X I_{[X \geq \lambda]} \geq y)\,dy \qquad \{\text{by (a)}$$

$$= \int_0^\lambda P(X I_{[X \geq \lambda]} \geq y)\,dy + \int_\lambda^\infty P(X I_{[X \geq \lambda]} \geq y)\,dy$$

show $\longrightarrow$ $\Updownarrow$ $\lambda P(X \geq \lambda)$ $\qquad$ $\Updownarrow$ $\int_\lambda^\infty P(X \geq y)\,dy$

$\downarrow$

using $\{\omega : X(\omega) I_{[X(\omega) \geq \lambda]} \geq y\} \equiv \{\omega : X(\omega) \geq \lambda\}$ when $y \in (0, \lambda)$

$\Rightarrow P\{ \quad \} = P\{ \quad \}$

and $\{\omega : X(\omega) I_{[X(\omega) \geq \lambda]} \geq y\} = \{X(\omega) \geq y\}$ when $y \in (\lambda, \infty)$

show equality of sets.

NOTE $E(X I_{[X \geq \lambda]}) > \int_\lambda^\infty P(X \geq t)\,dt$

(c) $Y \in \mathcal{L}_1$ st $P(|X_n| \geq y) \leq P(Y \geq y)$ $\forall y > 0$ & $n \geq 1$ Show $\{X_n\}$ are u.i

We must show $\sup_n \int X_n I_{[|X_n| \geq \lambda]} \longrightarrow 0$ as $\lambda \to \infty$ (def of u.i

Now $E(X_n I_{[|X_n| \geq \lambda]}) = \int_{[X_n \geq \lambda]} X_n \cdot dP$

$$= \lambda P(X_n \geq \lambda) + \int_\lambda^\infty P(X_n \geq y)\, dy \quad \{ \text{by part (b)}$$

$$\leq \lambda P(Y \geq \lambda) + \int_\lambda^\infty P(Y \geq y) \cdot dy$$

$$= \int_{[Y \geq \lambda]} Y \cdot dP$$

$$\leq \int Y \cdot dP$$

$$< \infty \qquad \text{since } Y \in \mathcal{L}_1$$

$\boxed{\# 3.5.6}$ a) $\xi \cong Unif(0,1)$ & $X_n = \left(\frac{n}{\log n}\right) I_{[0, 1/n]} \times \xi$ $\forall n \geq 3$

$$E(X_n) = \frac{n}{\log n} E\left(I_{[0,1/n]} \times \xi\right) = \frac{n}{\log n} \times \frac{1}{n} = \frac{1}{\log n} \longrightarrow 0$$

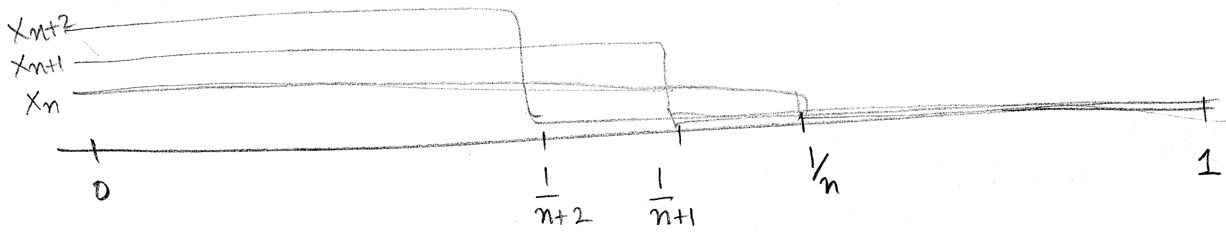
To show $X_n \xrightarrow{P} X$ ie $P(|X_n - X| > \varepsilon) \longrightarrow 0$

$$P\left(\left|\frac{n}{\log n} I_{[0,1/n]} \xi\right| > \varepsilon\right) \longrightarrow 0$$

$$P\left(\left|\frac{n}{\log n} I_{[0,1/n]}(\xi)\right| > \varepsilon\right) \overset{?}{\leq} P\left(|\xi \in [0, 1/n]|\right) = \frac{1}{n} \to 0$$

Now we have $X_n \xrightarrow{P} X$ & we can use Vitali's theorem.

$$E\,|X_n|^r \longrightarrow E\,|X|^r$$ use $r=1$, $X=0$ & we have done !!



looks like this by construct^n

let $\displaystyle Y \equiv \sum_{k=3}^{\infty} \frac{k}{\log k} I_{\left[\frac{1}{k+1}, \frac{1}{k}\right]}$ & $Y \geq X_n$

$$E(Y) = \sum_{k=3}^{\infty} \underbrace{\frac{1}{\log k}}_{b_n} \underbrace{\frac{1}{(k+1)}}_{a_n} = \infty$$

$\left\{ \begin{array}{l} \sum a_n b_n = bc \\ \text{If } \sum a_n \to c \ \& \sum b_n \to b \\ \& \sum \frac{1}{k+1} \Rightarrow \infty \end{array} \right.$

b) $\displaystyle Y_n \equiv n\, I_{[0,\frac{1}{n}]}(\xi) - n\, I_{[\frac{1}{n}, \frac{2}{n}]}(\xi)$

$$E(Y_n) = n\left(\tfrac{1}{n}\right) - n\left(\tfrac{1}{n}\right) = 1 - 1 = 0.$$

ie $\int Y_n\, dP \longrightarrow 0$

Now show Y_n are not u.i [If $A \Leftrightarrow B$ then $A^c \Leftrightarrow B^c$]

If we can show any cond^n of Vitali's is violated then we are done!

Now let $r=1$, $Y=0$ then show $Y_n \xrightarrow{P} 0$ (cond^n req by Vitali)

$$P(|Y_n| > \varepsilon) = P\left(\left|n I_{[0,1/n]}(\xi) - n I_{[\frac{1}{n},\frac{2}{n}]}(\xi)\right| > \varepsilon\right) \leq P\left(\xi \in (0, 2/n)\right) = \frac{2}{n} \to 0.$$

?

since Y_n can take only $-n, n, 0$.

If we can show $E|X_n| \not\to E|X|$.

Now $\cancel{E X_n}$ $|Y_n| = \left| n I_{[0,1/n]}(\omega) - n I_{[1/n, 2/n]}(\omega) \right|$

$$= \left| n I_{[0,1/n]} \right| + \left| n I_{(1/n, 2/n)} \right| \quad \Big\} \text{ disjoint supports}$$

$\omega \in$ either to $[0, \frac{1}{n}]$ $(\frac{1}{n}, 2/n]$ or $(2/n, 1]$ so check

$\therefore E|Y_n| = n\left(\frac{1}{n}\right) + n\left(\frac{1}{n}\right) = 2 \not\to 0$.

Exam 2

$X \geq 0$, $X \in \mathcal{L}_2$

$E(X) > a$, $a > 0$

Show: $$P(X>a) \geq \frac{[E(X)-a]^2}{E(X^2)}$$

Sol: $$\left[E\left(X\, I_{[X>a]}\right)\right]^2 \leq E(X^2)\, E\left(I_{[X>a]}\right)$$ Cauchy-Sc

Now $E\left(I_{[X>a]}\right) = P(X>a)$

$$\therefore \quad P(X>a) \geq \frac{\left[E\left(X\, I_{[X>a]}\right)\right]^2}{E(X^2)}$$

Now
$$E\left(X\, I_{[X>a]}\right) = \int X \cdot I_{[X>a]}\, d\mu$$
$$= \int_{[X>a]} X \cdot d\mu \pm \int_{[X<a]} X \cdot d\mu$$
$$= \int X \cdot d\mu - \int_{[X<a]} X \cdot d\mu$$
$$\geq \int X \cdot d\mu - \int_{[X<a]} a \cdot d\mu$$
$$\geq E(X) - a$$

$$E(X) = E\left(X I_{[X>a]}\right) + E\left[X I_{[X \leq a]}\right] \leq E\left(X\, I_{[X>a]}\right) + a$$
$$\Rightarrow \quad E(X) \leq E\left(X\, I_{[X>a]}\right) + a$$