Pay special attention to Chebyshev Ineq

FINALS REVIEW

FALL 05

9th Dec 05

#1. If $\sup_n \{ |X_n|^{1+\delta} \} < \infty$ for some $\delta > 0$.

Then $\{X_n\}$ is u.i.

#2 If

$$X_n = 0 \quad \text{wp} \quad 1 - \frac{1}{n^4}$$

$$= n^2 \quad \text{wp} \quad \frac{1}{n^4}$$

$$X = 0 \quad \text{wp} \quad 1$$

show

(1) $X_n \xrightarrow{P} X$

(2) $X_n \xrightarrow[as]{} X$

(3) $X_n \overset{r}{\not\longrightarrow} X$ for some $r > 0$.

#3

Suppose $\{X_n : n \geq 1\}$ converges in probability to X

ie $X_n \xrightarrow{P} X$

and $P(|X_n| > c) = 0 \quad \forall n \geq 1.$

To show: $X_n \xrightarrow{r} X$, where $r > 0$

1). To show $\sup_n E\left(|X_n| I_{[|X_n| \geq \lambda]}\right) \longrightarrow 0$ as $\lambda \to \infty$

Now $\int |X_n|^{1+\delta} dP \geq \int_{[|X_n| \geq \lambda]} |X_n|^{1+\delta} dP$

$$\geq \lambda^\delta \int_{[|X_n| \geq \lambda]} |X_n| dP \qquad \{ \text{since } \delta > 0$$

$$M > \sup_n \int |X_n|^{1+\delta} dP \geq \sup \left\{ \lambda^\delta \int_{[|X_n| \geq \lambda]} |X_n| . dP \right\}$$

$$\therefore \sup_n \left\{ \int_{[|X_n| \geq \lambda]} |X_n| dP \right\} \leq \frac{M}{\lambda^\delta}$$

$$\Rightarrow 0 \leq \sup_n \int_{[|X_n| \geq \lambda]} |X_n| dP \leq \frac{M}{\lambda^\delta} \longrightarrow 0 \quad \text{as } \lambda \to \infty$$

3). $P(|X_n| > c) = 0$ ie $|X_n| \leq c$ a.e

$X_n \xrightarrow{P} X$ ie $P(|X_n - X| > \varepsilon) \longrightarrow 0$

Now $|X| \leq |X_n - X| + |X_n|$

$$\therefore \left. \begin{array}{l} |X_n - X| < d \\ \& \\ |X_n| < a \end{array} \right\} \Rightarrow |X| \leq c + d$$

$$\therefore \underbrace{|X| > a+d}_{A} \Longrightarrow \underbrace{|X_n| \geq a}_{B} \text{ OR } \underbrace{|X_n - X| \geq d}_{C}$$

So $|X| > a+d \Rightarrow |X_n - X| \geq d$ OR $|X_n| > a$.

$$\therefore P(|X| > a+d) \leq \underset{\downarrow \atop 0}{P(|X_n - X| \geq d)} + \underset{\downarrow \atop 0}{P(|X_n| > a)}$$

$$\therefore P(|X| > a+d) = 0$$

$$|X| \leq a+d \quad wp\ 1$$

To show $E\,|X_n - X|^r \longrightarrow 0$ using $X_n \xrightarrow{P} X$, $|X_n| \leq c$, $|X| \leq c+d$.

We will use DCT

(i) Now $X_n \xrightarrow{P} X \Rightarrow |X_n - X| \xrightarrow{P} 0$ $\{ \because P(||X_n - X| - 0| > \varepsilon) = P(|X_n - X| > \varepsilon) \longrightarrow 0$

$\Rightarrow |X_n - X|^r \xrightarrow{P} 0$ $\{ X_n \xrightarrow{P} X \Rightarrow g(X_n) \xrightarrow{P} g(X)$ for $g \equiv$ cont.

(ii) Now To show $|X_n - X|^r \leq y$ for some $y \in \mathcal{L}_1$

Now C_r-inequality uses the fact $|x+y|^r \leq C_r\,|x|^r + C_r\,|y|^r$

$$\therefore |X_n - X|^r \leq C_r\,|X|^r + C_r\,|X_n|^r$$
$$< C_r\left[a^r + (a+d)^r\right]$$

Now let $y = C_r\left(a^r + (a+d)^r\right) \in \mathcal{L}_1$

since $\int |y|\,d\mu = \int |y|\,dP = C_r\left[a^r + (a+d)^r\right] \int dP < \infty$

(iii) Now by DCT $\left.\begin{matrix} |X_n - X|^r \leq y \\ |X_n - X|^r \xrightarrow{P} 0 \end{matrix}\right\} \Rightarrow \int |X_n - X|^r\,dP \longrightarrow 0$

ie $E(|X_n - X|^r) \longrightarrow 0$ as $n \to \infty$

ie $X_n \xrightarrow{\mathcal{L}_r} X$

2) $X_n = 0 \quad wp \quad 1 - 1/n^4$ $\qquad X = 0 \quad wp \quad 1$

$\quad = n^2 \quad wp \quad 1/n^4$

1) To show $X_n \xrightarrow{P} X$ ie $P(|X_n - X| > \varepsilon) \longrightarrow 0 \quad \forall\, \varepsilon > 0$

Consider $P(|X_n - X| > \varepsilon) = P(|X_n| > \varepsilon)$

$$= P(X_n = n^2) \qquad \text{since } \varepsilon > 0$$

$$= \frac{1}{n^4} \longrightarrow 0$$

3) To show $X_n \overset{\mathcal{L}_r}{\not\longrightarrow} X$ we must show $E|X_n - X|^r \not\longrightarrow 0$

ie $E(|X_n - X|^r) = E(|X_n|^r)$

$$= n^{2r} \times \frac{1}{n^4}$$

$$= n^{2r-4}$$

If $r \geq 2 \Rightarrow E(|X_n - X|^r) \not\longrightarrow 0$

2) To show $X_n \xrightarrow{a.s} X \Rightarrow$ for any k $P\left(\bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{m \geq n}^{\infty} |X_m - X| > 1/k\right) = 0$

Let $A_k = \left[\bigcap_{n=1}^{\infty} \bigcup_{m \geq n}^{\infty} |X_m - X| > 1/k\right]$

$\{A_k\} \uparrow$ seq

$$\therefore \quad P\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{N \to \infty} P\left(\bigcup_{k=1}^{N} A_k\right) \leq \lim_{N} \sum_{k=1}^{N} P(A_k)$$

Now $P(A_k) = P\Big(\bigcap_{n=1}^{\infty} \underbrace{\bigcup_{m \geq n}^{\infty} |X_m - X| \geq 1/k}_{\downarrow \text{ seq}}\Big) = \lim_{n \to \infty} P\left(\bigcup_{m \geq n}^{\infty} |X_m - X| \geq 1/k\right)$

$$\leq \lim_{n\to\infty} \sum_{m=n}^{\infty} P\left(|X_m - X| \geq 1/k\right)$$

$$= \lim_{n\to\infty} \sum_{m=n}^{\infty} P\left(|X_m| \geq 1/k\right)$$

$$= \lim_{n\to\infty} \sum_{m=n}^{\infty} \left(\frac{1}{m^4}\right)$$

$$= 0.$$

$$\therefore\ P(A_k) = 0$$

$$\therefore\ P\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} P(A_k) = 0.$$

To show $\xrightarrow{a.e}$ $\Leftrightarrow$ $P\left(\bigcap_{n=1}^{\infty} \bigcup_{m\geq n}^{\infty} |X_m - X| > \varepsilon\right) = 0 \quad \forall\ \varepsilon > 0$