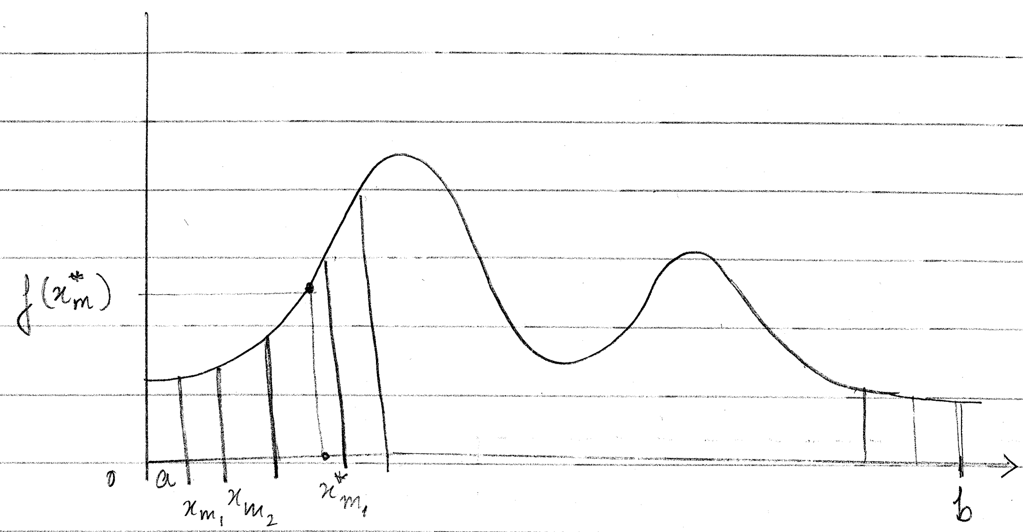
MOTIVATION



Rieman Integral [Divides the domain of the funct^n]

$f : [a,b] \longrightarrow R$, $f > 0$ f is continuous

$$RS_m = \sum f(x^*_{m_i})\,(x_{m_i} - x_{m,i-1}) \xrightarrow[n\to\infty]{} \int_a^b f(x)\,dx$$

$x^*_{m_i}$: some intermediate

This breaksdown if the funct^n f is discontinuous
If $f(\cdot)$ is discontinuous, identify the pts of discontinuity.
Put them together in a set and "measure" the size of this se

Eg: $f_n(x) = x^{n-1}$ where $f_n : [0,1] \longrightarrow R$ & $n = 1, 2 \ldots$

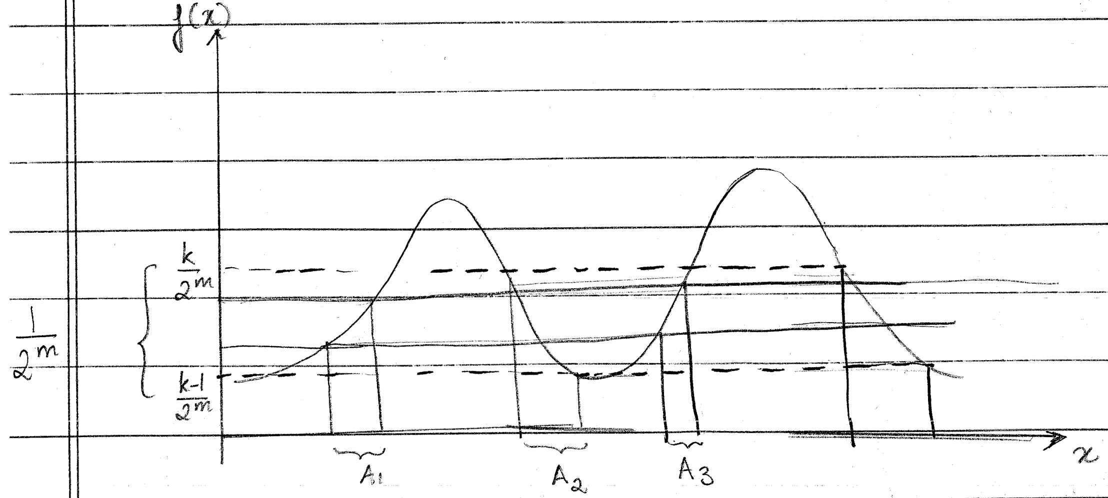
$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1 \end{cases} = f(x)$$

Does $\int_0^1 f_n(x)\,dx \longrightarrow \int_0^1 f(x)\,dx$

YES!

How do you measure area under the curve if the funct^n is discontinuous?

LEBESGUE MEASURE



Lebesgue thought that we should divide the range of the funct^n into intv of length $\frac{1}{2^m}$ say [m. intervals in all]

$$LS_m = \sum_{k=1}^{m2^m} \frac{k-1}{2^m} \times \text{measure}\left\{ x : \frac{k-1}{m} \leq f(x) \leq \frac{k}{2^m} \right\}$$

measure( ) = measure $\{A_1 \cup A_2 \cup A_3 \cup A_4\}$

We now need to define a measure and sets which can be measured.

PROB

Sample space Ω

A set funct^n $P : \Omega \rightarrow [0, 1]$

P is a prob if

(i) $P(\phi) = 0$ & $P(\Omega) = 1$

(ii) $\underbrace{P\left(\bigcup_{i=1}^{\infty} A_i\right)}_{P \text{ is countably additive}} = \sum_{i=1}^{\infty} P(A_i)$ if A_i's are d

$\mathcal{A}$ = collection of sets $\supset \bigcup_{i=1}^{\infty} A_i$ for $A_i \subseteq \Omega$

we need a set $\mathcal{A}$ which contains $\bigcup_{i=1}^{\infty} A_i$ and we actually de

$P : \mathcal{A} \longrightarrow [0, 1]$

We have to find a minimal $\mathcal{A} \subset \underbrace{2^{\Omega} = P(\Omega)}_{\text{all subsets of } \Omega}$

$\exists$ a $P : \mathcal{A} \rightarrow [0, 1]$ that satisfies (ii)

eg: Why only consider only countable unions.

$X \sim Unif[0,1]$

$P(x \in [0,1]) = 1$

$P(0 \leq X \leq 1) = \sum_{x \in [0,1]} P(X = x) = 0$

NOTATION

1 Ω : sample space

2 $2^{\Omega} = \mathcal{P}(\Omega)$: all possible subsets

3 $A \subset \Omega$: a subset / element of Ω

4 A^C : complement of set A

5 $A \cap B = AB$

6 $A \cup B = A + B$ if A & B are disjoint

7 $A \setminus B$ = diff of set A & B $= A \cap B^c$

8 $\{A_n\}_{n \geq 1}$ is an $\uparrow$ seq of sets if $A_n \subseteq A_{n+1}$ $\forall n$

9 A collection of sets is typically donated by curly letters.

DEF A class or a collection $\mathcal{A}$ of subsets A of some nonempty set Ω is called a SIGMA FIELD if

(a) If $A \in \mathcal{A}$ then $A^c \in \mathcal{A}$ [$\mathcal{A}$ is closed under complementa

(b) If $A_1, A_2 \ldots A_n \ldots$ is a countable collection of sets in $\mathcal{A}$
$\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$ [$\mathcal{A}$ is closed under countable

Note : Also known as σ-algebra

Note Based on the same Ω we can create many σ-fields.

2: $\mathcal{A}$ is called a field if $\mathcal{A}$ is closed under complements and finite unions.

Remark 1: If $\mathcal{A}$ is closed under complements and unions then $\mathcal{A}$ is also closed under intersections

$$\left(\bigcup_{n=1}^{\infty} A_n\right) \in \mathcal{A} \implies \left(\bigcup_{n=1}^{\infty} A_n\right)^c \in \mathcal{A}$$

$$\implies \bigcap_{n=1}^{\infty} A_n^c \in \mathcal{A}$$

$$\implies \bigcap_{n=1}^{\infty} B_n \in \mathcal{A} \qquad \text{where } B_n = A_n^c$$

$\therefore$ for any $\{B_n\}_{n\geq 1}$, $\bigcap_{n=1}^{\infty} B_n \in \mathcal{A}$

Remark 2: $\Omega = A \cup A^c$ for any $A \in \mathcal{A}$

$\therefore \Omega \in \mathcal{A}$ {prop ⓐ + ⓑ}

$\therefore \phi = \Omega^c \in \mathcal{A}$ {prop ⓐ}

PROPERTIES

1) Arbitrary intersections of σ-fields (/fields) are σ-fields (finite or countable)

HW 2) Countable unions of σ-fields is NOT necessarily a σ-field.

$\boxed{\text{DEF:}}$ let $\mathcal{C}$ be a class of subsets of Ω, $\sigma[\mathcal{C}]$ is the MINIMAL σ-field generated by $\mathcal{C}$ (that is containing $\mathcal{C}$)

Ex $\Omega = \{a, b, c, d, e\}$

$\mathcal{C} = \{E_1, E_2\}$ where $E_1 = \{a, b, c\}$

$E_2 = \{c, d\ e\}$

$$\sigma[\mathcal{C}] = \{\phi, F_1, F_2, F_3, F_1 \cap F_2, F_2 \cup F_3, \Omega\}$$

where $F_1 = \{a, b\} = E_1 \cap E_2^c$

$F_2 = \{d, e\} = E_2 \cap E_1^c$

$F_3 = \{c\} = E_1 \cap E_2$

$F_4 = \{a, b, d, e\} = F_1 \cup F_2$

* full

$\boxed{\text{DEF}}$ $\sigma[\mathcal{C}] = \bigcap_{\alpha} \{\mathcal{F}_\alpha : \mathcal{F}_\alpha$ is a σ-field of subsets of Ω for which

is called the σ-field generated by $\mathcal{C}$

$\boxed{\text{Prop.}}$ $\mathcal{A}$, is a set of subsets of Ω is a σ-field $\longleftrightarrow$ $\mathcal{A}$ is a field and a monotone class

$\boxed{\text{DEF}}$: $\mathcal{A}$ is a monotone class if $A_n \in \mathcal{A}\ \forall n$ where $\{A_n\}_n$ is $\uparrow$ then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$

and if $\{A_n\}_n$ is a $\downarrow$ seq then $\bigcap_{n=1}^{\infty} A_n \in \mathcal{A}$

Proof $\Leftarrow$ $A_1 \cup A_2 \cup A_n \cdots \cup \cdots = \bigcup_{n=1}^{\infty} A_n$

$$= \bigcup_{n=1}^{\infty} \left[\bigcup_{k=1}^{n} A_k\right] = \bigcup_{n=1}^{\infty} [B_n]$$

Now $B_n = \bigcup_{k=1}^{n} A_k \in \mathcal{A}$ {since $\mathcal{A}$ is a field}

Also $\{B_n\}$ is an $\uparrow$ seq $\therefore$ $\bigcup_{n=1}^{\infty} \{B_n\} \in \mathcal{A}$ {since $\mathcal{A}$ is a monotone class}

F

Let $\underline{\mathcal{M} : \mathcal{A} \to [0, \infty)}$ be a set function st $\underline{\mu(\phi) = 0}$
Let $\mathcal{A}$ be a σ-field (does not have to be a σ-field)
μ is called a <u>countably additive measure</u> if

$$\boxed{\mu\left(\sum_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)}$$

<u>Remark</u> : We can still define μ on a field
An example of a measure $\to$ length of an interval

ION

$\underline{[\Omega, \mathcal{A}, \mu]}$ is called a <u>measure space</u>

A probability measure is a particular kind of measure where the image of the function is $[0, 1]$

What happens if we do not have disjoint unions

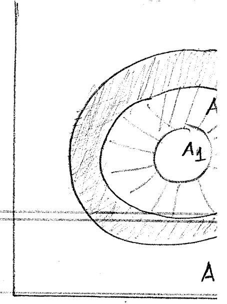
S.

<u>Monotone Property of Measures</u>

Let $(\Omega, \mathcal{A}, \mathcal{M})$ be a measure space

(a) $$\boxed{\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} \mu(A_n)}$$

when $\underline{\{A_n\}_n \text{ be an } \uparrow \text{ seq}}$ of events ie $A_n \subseteq A_{n+1} \ \forall n$



$$\boxed{\text{Notation: If } \{A_n\}_n \text{ is } \uparrow \text{ then } \lim_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} A_n}$$

Proof

$$\bigcup_{n=1}^{\infty} A_n = A_1 \cup A_2 \cup A_3 \ldots$$

$$= \underbrace{(A_1 \setminus A_0) \cup (A_2 \setminus A_1) \cup \ldots \cup (A_{n+1} \setminus A_n)}_{\text{disjoint}} \cup \ldots$$

$$= \sum_{n=1}^{\infty} (A_{n+1} \setminus A_n)$$

$$\therefore \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\sum_{n=1}^{\infty} (A_{n+1} \setminus A_n)\right)$$

$$= \sum_{n=1}^{\infty} \mu(A_{n+1} \setminus A_n)$$

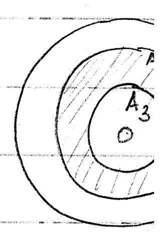
$$= \lim_{n\to\infty} \left[\sum_{k=1}^{n} \mu(A_{k+1} \setminus A_k)\right]$$

$$= \lim_{n\to\infty} \left(\sum_{k=1}^{n} [\mu(A_{k+1}) - \mu(A_k)]\right)$$

$$= \lim_{n\to\infty} \mu(A_n)$$

(b) Assume that μ is finite. Let $A_n \supset A_{n+1}$ $\forall n$ ie $\{A_n\}_n$
then

$$\boxed{\mu\left[\bigcap_{n=1}^{\infty} A_n\right] = \lim_{n\to\infty} \mu(A_n)}$$



Proof

Let $B_n = A_1 \setminus A_n \quad \forall n$

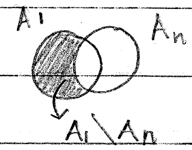
$\mu(B_n) = \mu(A_1) - \mu(A_n) \quad \forall n$

$$\lim_{n\to\infty} \mu(B_n) = \mu(A_1) - \lim_{n\to\infty} \mu(A_n)$$

Now $\{B_n\}_n$ is an increasing seq

$$\therefore \lim_{n\to\infty} \mu(B_n) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) \qquad \{\text{using (a) \& Notation}$$

$$= \mu\left(\bigcup_{n=1}^{\infty} (A_1 \setminus A_n)\right)$$



$$= \mu\left[\bigcup_{n=1}^{\infty} A_1 \cap A_n^c\right]$$

$$= \mu\left[A_1 \cap \left(\bigcup_{n=1}^{\infty} A_n^c\right)\right]$$

$$= \mu\left[A_1 \cap \underbrace{\left(\bigcap_{n=1}^{\infty} A_n\right)^c}\right]$$

$\longrightarrow$ subset of A_1 & $[\;] = A_1 \setminus (\cap A_n)$

$$= \mu(A_1) - \mu\left(\bigcap_{n=1}^{\infty} A_n\right)$$

$$\therefore \mu(A_1) - \mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \mu(A_1) - \lim_{n\to\infty} \mu(A_n)$$

$$\therefore \mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n)$$

(c) $\boxed{\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty} \mu(A_n)}$ for <u>any</u> $\{A_n\}_n$

Called "<u>Countable</u> <u>subadditivity</u>"

Claim $\mathcal{B} \subset 2^{\mathbb{R}}$

MEASURES ON $\mathbb{R}$

DEF $A \subset \mathbb{R}$; Lebesgue measure $\lambda(A) =$ length of A

Ques $\lambda : (\mathbb{R}, 2^{\mathbb{R}}) \longrightarrow [0, \infty)$ can you measure any possible subset of $2^{\mathbb{R}}$

No!

What kind of sets can you measure then?

Let $I = \{ (a,b], (-\infty, b], (a, \infty) : \text{for any } a, b \in \mathbb{R} \}$

I is NOT a field

Let $\mathcal{B}_I$ be a collection of sets consisting of all finite disjoint union sets in I [$\mathcal{B}_I$ is a field, can be shown]

DEF $\mathcal{B} = \sigma[\mathcal{B}_I]$ is a σ-field generated by $\mathcal{B}_I$ [$\mathcal{B}_I$ = generator]

$\mathcal{B}$ is called a Borel set.

Alt Char $\mathcal{B} = \sigma[\underbrace{\mathcal{A}}_{\text{family of all open intv in } \mathbb{R}}] = \sigma[\underbrace{\mathcal{C}}_{\text{family of all closed intv in } \mathbb{R}}]$

Aside : $\mu : (\Omega, \mathcal{A}) \longrightarrow [0, \infty)$ & μ is countably additive

Now let $A \in \mathcal{A}$, $\delta_{x_0}(A) = \begin{cases} 1 & \text{if } x_0 \in A \\ 0 & \text{ow} \end{cases}$ where x_0 is a element of Ω

$\delta_{x_0}(A)$ is a Dirac measure [measure concentrated at a point] is also an instance of a measure

PROP $\mathcal{B}_I$ is a field

DEF : Let $\lambda : (\mathbb{R}, \mathcal{B}_I) \longrightarrow [0, \infty)$

$$\lambda(A) = \sum_{j=1}^{J_0} \lambda(A_j) \qquad \text{where } \lambda \text{ is length}$$

For any $A \in \mathcal{B}_I$, $A = \sum_{j=1}^{J_0} A_j$, $A_j \in \mathcal{B}_I$

Note : You do not need to write $\lambda : (\mathbb{R}, \mathcal{B}_I) \to [0,\infty)$ you can also write
$\lambda : (\mathcal{B}_I) \to [0,\infty)$ $\mathbb{R}$ is included as a reminder.

Then λ is a measure on the field $\mathcal{B}_I$ and we call λ the Lebesgue measure

M **The Carathiodory Extension theorem:**

A (σ-finite) measure defined on a field $\mathcal{C}$ has a unique extension to the generated σ-field, $\sigma[\mathcal{C}]$

of req

Thus by the Extension theorem we can extend the Lebesgue measure λ (defined on $\mathcal{B}_I$) to a measure on $\mathcal{B}$. We still call this extended measure λ

Overview

Start with a measure λ = length on $\mathbb{R}$

$\downarrow$

$(\mathbb{R}, \mathcal{B}_I)$ we defined Lebesgue measure on $\mathcal{B}_I$ (a field) ; λ

$\downarrow$ using theorem 1

$(\mathbb{R}, \mathcal{B})$ we got the Lebesgue measure on $\mathcal{B}$; λ

Is $\mathcal{B} \subset 2^{\mathbb{R}}$? YES

[Can I use λ to measure all elements on $2^{\mathbb{R}}$?]
[Can you define a set funct^n on $2^{\mathbb{R}}$ such that it is a Lebesgue measure ?] NO
$\lambda : (\mathbb{R}, 2^{\mathbb{R}}) \to [0,\infty)$

Proof by contradiction on Pg 15 of notes [Lec 3 - Outer measures ; Ext theorem]

λ the Lebesgue measure is related to the uniform dist^n

What does $\mathcal{B}$ contain ?

(i) All types of intervals
(ii) All singletons
(iii) All finite sets
(iv) Any countable set $(\mathbb{N}, \mathbb{Z}, \mathbb{Q})$
(v) $\mathbb{R} \setminus \mathbb{Q}$ set of irrationals and a lot more.

$\boxed{\text{PROP}}$. The Lesbesgue measure of any countable set is always zero !

$$\lambda(\mathbb{Q}) = \sum_{q \in \mathbb{Q}} \lambda\{q\} = \sum_{q \in \mathbb{Q}} 0 = 0 \quad \text{since } \lambda\{q\} = 0$$

The cardinality of a set has nothing to do with its lebesgue measure

Example : A uncountable set which has lebesgue measure zero
cantor set :

Start with $[0, 1]$
Remove the middle $1/3$ ie $(1/3, 2/3)$
Remove the middle $1/3$ of each of the parts ie $(1/9, 2/9)$ & (
⋮

What is left over is called a Cantor set C

$C \in \mathcal{B}$ and $\lambda(C) = 0$

MORE GENERAL MEASURES ON $\mathbb{R}$

A measure μ (set funct) on $\mathbb{R}$, assigning finite values to finite intervals is called a Lebesgue - Stieltjes measure (L-S measure)

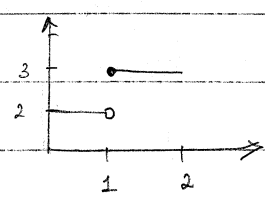
eg: the Lebesgue measure is an instance of the Lebesgue - Stieltjes measure.

Let F be a finite valued function on $\mathbb{R}$. If
(i) F is increasing
(ii) F is right continuous $\left\{ \lim_{\substack{x \to x_0 \\ x > x_0}} g(x) = g(x_0) \Rightarrow g \text{ is rt continuous at } x_0 \right\}$
then F is called a generalized dist^n function

Def: g is right continuous at x_0 if $\lim_{\substack{x \to x_0 \\ x > x_0}} g(x) = g(x_0)$

Def: g is left continuous at x_0 if $\lim_{\substack{x \to x_0 \\ x < x_0}} g(x) = g(x_0)$

eg: $g(x) = \begin{cases} 2 & x \in (0,1) \\ 3 & x \in [1,2) \end{cases}$



F is a finite valued funct^n on $\mathbb{R}$ such that
F is increasing
F is right continuous
$F(0) = 0$
is called a representative generalized dist^n function

THEOREM

Correspondence Theorem

Let F be a finite valued funct^n on $\mathbb{R}$ st F is $\uparrow$, right continuou
and $F(0) = 0$

Let μ be a L-S measure on $(\mathbb{R}, \mathcal{B})$

then

$$\boxed{\mu((a,b]) = F(b) - F(a)} \quad ; \text{for} \quad -\infty \leq a < b \leq \infty$$

establishes a 1-1 correspondence b/w the L-S measure on $\mathbb{R}$ and the generalized representative distribution function.

Example 1) If $F(x) = x$ then $\mu = \lambda$ (lebesgue measure)

2) $F(x) = \ ?$ st $\mu = \delta_{x_0}((a,b])$ for a given x_0

$$F(x) = 1_{[x_0, \infty)}(x)$$

$$= \begin{cases} 1 & \text{if } x \geq x_0 \\ 0 & \text{ow} \end{cases}$$

Proof

We have μ, construct F satisfying the req.

Define $F(x) = \begin{cases} \mu(\{0\}) - \mu((x,0]) & \text{for } x < 0 \\ \mu(\{0\}) & \text{for } x = 0 \\ \mu(\{0\}) + \mu((0,x]) & \text{for } x > 0 \end{cases}$

[Note : If μ was λ the Lebesgue measure then $\mu(\{0\}) = ($
But here $\mu = \delta_{\{0\}} \Rightarrow \mu(\{0\}) = \delta_{\{0\}}(\{0\}) = 1$ infact

Since $\mu(\{0\})$ and $\mu((x,0])$ are +ve quantities, by construction F is

f is contin. at $x_0 \iff \lim_{n\to\infty} f(x_n) = f(x_0) \quad \forall\ x_n \to x_0$

f is right continuous at $x_0 \iff \lim_{n\to\infty} f(x_n) = f(x_0) \quad \forall\ x_n \downarrow x_0$

To show $\lim_{\substack{x\to 0\\ x>0}} F(x) = F(0) = \mu(\{0\})$ for right continuity.

Let $a_n \downarrow 0$ ie $(a_1 \geq a_2 \geq \cdots \geq a_n \cdots > 0)$

Now $\lim_{n\to\infty} F(a_n) = \mu(\{0\}) + \lim_{n\to\infty} \mu((0, a_n])$ $\{$since $a_n > 0$

$$= \mu(\{0\}) + \mu\left(\bigcap_{n=1}^{\infty} (0, a_n]\right) \quad \{\text{by proposition}\}$$

$$= \mu(\{0\}) + \mu(\phi)$$

$$= \mu(\{0\})$$

$\Rightarrow F(0)$ is right continuous

Now given the F can we construct a μ

Define $\mu((a,b]) = F(b) - F(a)$

Since F is finite valued $\rightarrow \mu$ is finite for finite intv

We want μ on $\mathcal{B} = \sigma[\mathcal{C}_F]$

$\mathcal{C}_F$ = all finite disjoint unions from $\mathcal{C}_I$

$\mathcal{C}_I$ = all intervals $(a,b]$ $(-\infty, b]$ (a, ∞)

Let I be the collection of all finite intervals of type $(a,b]$

$$(a,\infty) = \bigcup_{n=1}^{\infty} (a,n] = \sum_{n=1}^{\infty} J_n \longrightarrow \text{sum of disjoint intervals}$$

$$(-\infty, b] = \bigcup_{n=1}^{\infty} (-n, b] = \sum_{n=1}^{\infty} J_n'$$

We can use carathédory ext them & show it is a measure on only $\mathcal{C}_F$ which in turn means we only need to prove that μ is a measure on finite intervals

Assume $(a,b] = \sum_{n=1}^{\infty} (a_n, b_n]$ { $(a,b]$ can be written as disjoint union of pieces

$$= \sum_{n=1}^{\infty} I_n$$

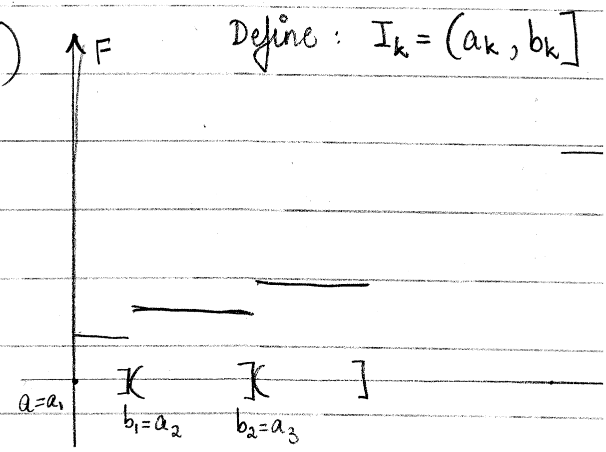
By definition $\mu((a,b]) = F(b) - F(a)$

To show $\mu\left(\sum_{n=1}^{\infty} I_n\right) = \sum_{n=1}^{\infty} \mu(I_n) = \mu((a,b])$

and we will show it holds for any decomposition of $(a,b]$ in disjoint intervals

ⓐ To show $\sum_{n=1}^{\infty} \mu(I_n) \leqslant \mu\left(\sum_{n=1}^{\infty} I_n\right) = \mu((a,b])$

Define : $I_k = (a_k, b_k]$



Let n be fixed

$$\sum_{k=1}^{n} I_k \subset (a,b]$$

(*) $\lim_{\varepsilon \to 0} F(a+\varepsilon) = F(a)$ since $F(\cdot)$ is right continuous

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$\sum_{k=1}^{n} \mu(I_k) = \sum_{k=1}^{n} \left(F(b_k) - F(a_k)\right)$ { by construction

$\leqslant F(b) - F(a)$ { since F is $\uparrow$ $\sum_1^n F(a_k, b_k] \leq F(a,b]$

$\lim_{n\to\infty} \sum_{k=1}^{n} \mu(I_k) = \lim_{n\to\infty} \sum_{k=1}^{n} \left(F(b_k) - F(a_k)\right) \leqslant F((a,b])$

ie $\sum_{k=1}^{\infty} \mu(I_k) \leq \mu((a,b])$ ——— (*)

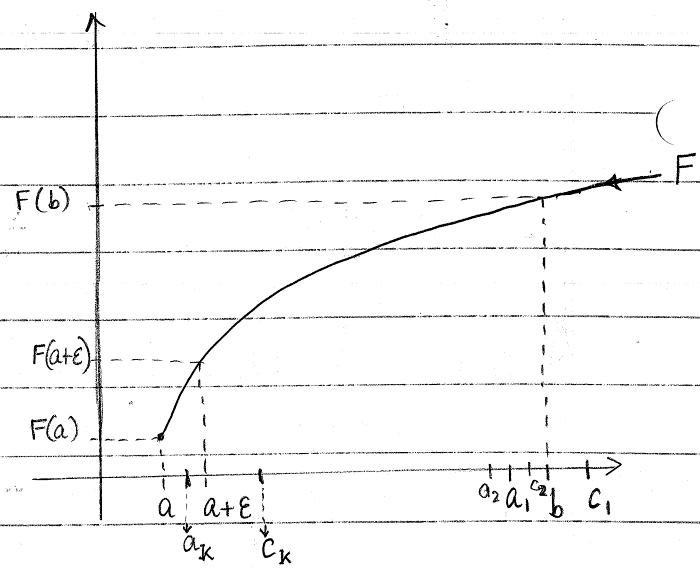
(b) To show $\mu((a,b]) = \mu\left(\sum_{n=1}^{\infty} I_n\right) \leqslant \sum_{n=1}^{\infty} \mu(I_n)$

ie $F(b) - F(a) \leqslant \sum_{n=1}^{\infty} \mu(I_n)$

This reduces to showing :

$$F(b) - F(a+\varepsilon) \leqslant \sum_{n=1}^{\infty} \mu(I_n) + \varepsilon \quad \forall \varepsilon > 0$$

Remarks : HeineBorel theorem



$[a+\varepsilon, b] \subset (a,b]$

$= \sum_{n=1}^{\infty} (a_n, b_n]$

$\subseteq \underbrace{\sum_{n=1}^{\infty} (a_n, b_n+\varepsilon)}_{\text{open cover}}$, $\forall \varepsilon > 0$

If we have a compact interval coverd by a ∞ union of open sets then $\exists$ a finite cover $\{(a_k, c_k)\}_{k=1}^{K}$ of $[a+\varepsilon, b]$

Let $\quad c_n = b_n + \varepsilon_n \quad , \quad c_k = b_k + \varepsilon_k$

$$[a+\varepsilon, b] \subset \bigcup_{k=1}^{K} (a_k, c_k)$$

?

$$F(b) - F(a+\varepsilon) \leq F(c_1) - F(a_K) \overset{\text{show}}{\leq} \sum_{k=1}^{K} (F(c_k) - F(a_k)$$

$$= \sum_{k=1}^{K} \left[F(c_k) - F(b_k) + F(b_k) - \right.$$

$$= \sum_{k=1}^{K} \left[\mu(I_k) + F(c_k) - F(b_k) \right.$$

$$= \sum_{k=1}^{K} \mu(I_k) + \sum_{k=1}^{K} \left[F(c_k) - F(b \right.$$

$$\therefore \quad F(b) - F(a+\varepsilon) \leq \sum_{k=1}^{K} \mu(I_k) + \sum_{k=1}^{K} \left[F(c_k) - F(b_k) \right]$$

We now select ε_k st $\quad F(c_k) - F(b_k) = F(b_k + \varepsilon_k) - F(b_k) < \frac{\varepsilon}{2^k}$

possible by right cont of

$$\therefore \quad F(b) - F(a+\varepsilon) \leq \sum_{k=1}^{K} \mu(I_k) + \sum_{k=1}^{K} \frac{\varepsilon}{2^k}$$

$$F(b) - F(a+\varepsilon) \leq \sum_{k=1}^{\infty} \mu(I_k) + \frac{\varepsilon}{2} \sum_{k=1}^{\infty} \left(\frac{1}{2^k} \right) \quad \left\{ \text{Taking li} \atop K- \right.$$

$$= \sum_{k=1}^{\infty} \mu(I_k) + \varepsilon$$

$$\lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} F(b) - F(a+\varepsilon) = F(b) - F(a) \quad \left\{ \text{By right continuity of } F(\cdot) \right.$$

$$\therefore \lim_{\varepsilon\to 0} F(b)-F(a+\varepsilon) \leq \lim_{\varepsilon\to 0} \sum_{k=1}^{\infty} \mu(I_k) + \varepsilon$$

$$\therefore \quad F(b)-F(a) \leq \sum_{k=1}^{\infty} \mu(I_k) \qquad \text{———} (**)$$

From $(*)$ & $(**)$ $\Longrightarrow$ $\mu((a,b]) = \mu\left(\sum_{k=1}^{\infty} I_k\right) = \sum_{k=1}^{\infty} \mu(I_k)$

$\therefore$ μ is a countably additive measure on $\mathcal{C}_F$

We now need to show it is well defined

ie given a set $A = \sum_{i=1}^{\infty} I_n$ OR $A = \sum_{m=1}^{\infty} I_m$ (write it as a union of disjoint intervals)

then $\mu(A) = \mu\left(\sum_{n=1}^{\infty} I_n\right) = \sum_{n=1}^{\infty} \mu(I_n)$

and $\mu(A) = \mu\left(\sum_{m'=1}^{\infty} I_m'\right) = \sum_{m'=1}^{\infty} \mu(I_m')$

we must show $\sum_{n=1}^{\infty} \mu(I_n) = \sum_{m=1}^{\infty} (I_m')$.

Now notice $I_m' = A \cap I_m'' = \left(\sum_{n=1}^{\infty} I_n\right) \cap I_m' = \sum_{n=1}^{\infty} (I_n I_m')$

$I_n = A \cap I_n = \left(\sum_{m=1}^{\infty} I_m'\right) \cap I_n = \sum_{n=1}^{\infty} (I_m' I_n)$

Now $\sum_{m=1}^{\infty} \mu(I_m') = \sum_{m=1}^{\infty} \mu\left(\sum_{n=1}^{\infty} I_m' I_n\right)$

$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \mu(I_n I_m')$

$= \sum_{n=1}^{\infty} \mu\left(\sum_{m=1}^{\infty} I_n I_m'\right) = \sum_{n=1}^{\infty} \mu(I_n)$

QED

MEASURABLE FUNCTIONS

* { $\mathcal{A}$ & $\mathcal{A}'$ are σ-fields on Ω }

DEF Let $X : \Omega \longrightarrow \Omega'$ be a function (arbitrary).
Let $(\Omega, \mathcal{A})$ and $(\Omega', \mathcal{A}')$ be 2 measure spaces *
X is called measurable if $X^{-1}(\mathcal{A}') \subset \mathcal{A}$
Note : X is called $\mathcal{A}'$-$\mathcal{A}$ measurable.
Notation : Inv image of a set via a funct^n
$A \in \mathcal{A}'$
$X^{-1}(A) = \{ \omega \in \Omega \mid X(\omega) \in A \}$ for sets
$X^{-1}(\mathcal{A}') = \{ X^{-1}(A), A \in \mathcal{A}' \}$ for sets of sets

PROP Let $X : (\Omega, \mathcal{A}) \longrightarrow (\mathbb{R}, \mathcal{B})$ be an arbitrary function
Suppose $\mathcal{B} = \sigma[\mathcal{C}]$

(1) X is measurable iff $X^{-1}(\mathcal{C}) \subset \mathcal{A}$
(2) X is measurable iff $X^{-1}((-\infty, x]) \in \mathcal{A} \quad \forall x \in \mathbb{R}$

Proof (Reading Ex - Pg 21-23)

We know $X^{-1}(\sigma[\mathcal{C}]) = \sigma[X^{-1}(\mathcal{C})]$ for any $\mathcal{C} \subset \mathcal{B}$

$\Leftarrow$ We need to show X is measurable ie show $X^{-1}(\mathcal{B}) \subset \mathcal{A}$

$\mathcal{C}_1 \subset \mathcal{C}_2$
$\sigma[\mathcal{C}_1] \subset \sigma[\mathcal{C}_2]$

Now $X^{-1}(\mathcal{B}) = X^{-1}(\sigma[\mathcal{C}]) = \sigma[X^{-1}(\mathcal{C})]$ } $\Rightarrow X^{-1}(\mathcal{B}) \subset \sigma$
Also $X^{-1}(\mathcal{C}) \subset \mathcal{A}$ }

$\Rightarrow X^{-1}(\mathcal{B}) \subset \mathcal{A}$

{ since $\mathcal{A}$ is a σ-field $\Rightarrow \sigma[\mathcal{A}] = \mathcal{A}$ }

Class of measurable functions is larger than the class of continuous functions.

(2) Take $\mathcal{C} = \mathcal{C}_I = \{(a,b], (-\infty,b], (a,\infty) : a,b \in \mathbb{R}\}$

If $(-\infty, x]$ generates $\mathcal{C}$ we are done!

$$(a,b] = (-\infty, b] \cap (-\infty, a]^c$$
$$(-\infty, b) = \bigcup_{n=1}^{\infty} (-\infty, b - 1/n]$$

Read Prop 2.2 – Pg 25

Ex 1 If X is measurable, then cX $(c>0)$ is also measurable

Proof Let $Z = cX$ & we know $X : (\Omega, \mathcal{A}) \rightarrow (\mathbb{R}, \mathcal{B})$

Consider
$$Z^{-1}((-\infty, x]) = \{\omega \in \Omega : Z(\omega) \in (-\infty, x]\}$$
$$= \{\omega \in \Omega : Z(\omega) \leq x\}$$
$$= \{\omega \in \Omega : Z(\omega) \leq x\}$$
$$= \{\omega \in \Omega : cX(\omega) \leq x\}$$
$$= \{\omega \in \Omega : X(\omega) \leq x/c\}$$
$$= X^{-1}((-\infty, x/c]) \in \mathcal{A}$$

Ex 2 If X is measurable ang 'g' is continuous funct^n then the composite function $g(X)$ is measurable.

Proof

Step 1: Assume 'g' is measurable and prove the result

Step 2: Show that any continuous funct^n is measurable

$$g(X) = g \circ X$$

We must show $(g \circ X)^{-1}(\mathcal{B}) \subset \mathcal{A}$

Now $(g \circ X)^{-1}(\mathcal{B}) = (X^{-1} \circ g^{-1})(\mathcal{B}) = X^{-1}(g^{-1}(\mathcal{B}))$

$$\begin{tikzcd} (\Omega, \mathcal{A}) \arrow[r, "X"] \arrow[dr, "g \circ X"'] & (\mathbb{R}, \mathcal{B}) \arrow[d, "g"] \\ & (\mathbb{R}, \mathcal{B}) \end{tikzcd}$$

Now assume g is measurable, $g^{-1}(\mathcal{B}) \subset \mathcal{B}$.

$\mathcal{A}_1 = X^{-1}(g^{-1}(\mathcal{B})) = \{ X^{-1}(D) : D \in g^{-1}(\mathcal{B}) \}$

$\mathcal{A}_2 = X^{-1}(\mathcal{B}) = \{ X^{-1}(C) : C \in (\mathcal{B}) \}$

To show $\mathcal{A}_1 \subset \mathcal{A}_2$

Let $F \equiv X^{-1}(D)$ st $D \in g^{-1}(\mathcal{B})$ ie $F \in \mathcal{A}_1$

We want to show $F \in \mathcal{A}_2$

$$D \in g^{-1}(\mathcal{B}) \subset \mathcal{B} \Rightarrow D \in \mathcal{B}$$

$$\therefore (g \circ X)^{-1}(\mathcal{B}) = X^{-1}(g^{-1}(\mathcal{B})) \subset X^{-1}(\mathcal{B}) \subset \mathcal{A}$$

Step 2: Now g is continuous

We want to show $g^{-1}(\mathcal{B}) \subset \mathcal{B}$

$$\begin{aligned} g^{-1}(\mathcal{B}) &= g^{-1}(\sigma[\text{open sets}]) \\ &= \sigma[g^{-1}(\text{open sets})] \\ &\subset \sigma[\text{open sets}] = \mathcal{B} \end{aligned}$$

$\{ g^{-1}(\text{open sets}) \equiv \text{open set}$

DEF

Simple function: Given a measure space $(\Omega, \mathcal{A})$

$$I_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$$

A simple function is a piecewise funct^n $X : (\Omega, \mathcal{A}) \rightarrow (\overline{\mathbb{R}}, \overline{\mathcal{B}}$

$$X(\omega) = \sum_{i=1}^{n} x_i I_{A_i}(\omega) \quad \text{where } \sum_{i=1}^{n} A_i = \Omega$$

$A_i \in \mathcal{A}$ and x_i's $\in$

NOTE This function is measurable but not continuous

$X : \Omega \to \bar{\mathbb{R}}$ is measurable $\iff$ X is the limit of a seq of simple funct^n

(Note this convergence has to only be pointwise)

Proof "$\Leftarrow$" Pg 25 Prop 2.2 $\lim_{n\to\infty} X_n$ is measurable if $\{X_n\}_n$ is measurable

"$\Rightarrow$" Let X be given (and it is measurement)

We define

$$X_n = \sum_{k=1}^{n \cdot 2^n} \frac{k-1}{2^n} \left\{ I_{\left[\frac{k-1}{2^n} \leq X < \frac{k}{2^n}\right]} - I_{\left[\frac{k-1}{2^n} \leq -X < \frac{k}{2^n}\right]} \right\}$$

$$+ \; n \left\{ I_{[X \geq n]} - I_{[-X \geq n]} \right\}$$

Notice:

$$\mathcal{C} = \left\{ \left\{ \omega \in \Omega \,\middle|\, \frac{k-1}{2^n} \leq X(\omega) < \frac{k}{2^n} \right\}_{k=1}^{n2^n}, \left\{ \omega \in \Omega \,\middle|\, \frac{k-1}{2^n} \leq -X(\omega) < \frac{k}{2^n} \right\}, [X \geq n], [X \leq -n] \right\}$$

consists of disjoint intervals whose union is Ω

Now $[X \geq n] = \{\omega \in \Omega \mid X(\omega) \geq n\} = \{\omega \in \Omega \mid X(\omega) \in [n, \infty)\} = X^{-1}([n,\infty))$

Now since X is measurable $X^{-1}(\mathcal{B}) \subset \mathcal{A}$

$\Rightarrow$ each of the sets in $\mathcal{C}$ $\in \mathcal{A}$ since $\mathcal{B}$ can be generated by any of the sets in $\mathcal{B}$.

Each X_n is a simple function by construction.

We need to show $\lim_{n\to\infty} X_n(\omega) = X(\omega) \quad \forall \quad \omega \in \Omega$ (pt-wise

Let $\omega_0 \in \Omega$ be fixed arbitrary

X is unbdd

Case 1 : $|X(\omega_0)| \geq n \Rightarrow X(\omega_0) \geq n$ or $X(\omega_0) \leq -n$

$$\Rightarrow X_n(\omega_0) = \begin{cases} n & \text{if } X(\omega_0) \geq n \\ -n & \text{if } X(\omega_0) \leq -n \end{cases}$$

$$\lim_{n\to\infty} X_n(\omega_0) = \begin{cases} \infty \\ -\infty \end{cases}$$

Also if $X(\omega_0) \geq n \quad \forall n \Rightarrow X(\omega_0) = \infty$

$X(\omega_0) \leq -n \quad \forall n \Rightarrow X(\omega_0) = -\infty$

$$\therefore \lim_{n\to\infty} X_n(\omega_0) = X(\omega_0)$$

Case 2 $\quad |X(\omega)| \leq n \Rightarrow -n \leq X(\omega) \leq n$

X is bdd

It is enough to show pt wise convergence for $0 \leq X(\omega_0)$ since the other one follows by symmetry.

$$\therefore X_n(\omega_0) = \sum_{k=1}^{n2^n} \frac{k-1}{2^n} I_{\left[\frac{k-1}{2^n} \leq X(\omega) \leq \frac{k}{2^n}\right]}(\omega_0)$$ where $X(\omega_0) \in$

This is like convg of seq of #'s $\lim_{n\to\infty} X_n(\omega_0) = X_n(\omega_0)$

Now $X(\omega_0) = X(\omega_0) \cdot 1 = X(\omega_0) \sum_{k=1}^{n2^n} I_{\left[\frac{k-1}{2^n} \leq X(\omega) \leq \frac{k}{2^n}\right]}(\omega_0)$

(*) $\left|\sum_n a_n\right| \leq \sum_n |a_n|$

since $[0, n] = \sum_{k=0}^{n2^n} \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right]$

$$\left|X_n(w_0) - X(w_0)\right| \leqslant \sum_{k=1}^{n2^n}{}' \left|\frac{k-1}{2^n} - X(w_0)\right| I_{\left[\frac{k-1}{2^n} \leq X(w) \leq \frac{k}{2^n}\right]}(w_0)$$

$$\leqslant \frac{1}{2^n}$$

since only one term in the sum is nonzero for the interval to which $X(w_0)$ belongs. Also the max possible abs value is the length of the interval to which $X(w_0)$ belongs, and each of the intervals is of length $\frac{1}{2^n}$

$\therefore \quad \left|X_n(w_0) - X(w_0)\right| \leqslant \frac{1}{2^n} \longrightarrow 0$ as $n \to \infty$

$\therefore \quad \lim_{n\to\infty} X_n(w_0) = X(w_0)$

RProj.org — Toolbox for MCMC
Matlab

** Bayes & Empirical Bayes Methods for Data Analysis
— Bradley Carlin & Thomas Louis

CONVERGENCE

$m \wedge n \equiv \min\{m, n\}$

$\forall \longleftrightarrow \cap$

$\exists \longleftrightarrow \cup$

DEF ## Convergence Almost Everywhere

Let $X_1, X_2, \ldots X_n, \ldots$ be a seq of functions

$$X_n : (\Omega, \mathcal{A}, \mu) \longrightarrow (\overline{\mathbb{R}}, \overline{\mathcal{B}})$$

We say $X_n \xrightarrow[n\to\infty]{a.e} X$ if $X_n(\omega) \xrightarrow[n\to\infty]{} X(\omega)$ $\forall\, \omega \in \Omega$ except $\omega \in N$ for some with $\mu(N) = 0$

PROP 1 $X_n \xrightarrow[n\to\infty]{a.e.} X \iff X_n - X_m \xrightarrow[m \wedge n]{a.e} 0$ {Cauchy convg a.…

Terminology $\{X_n\}_n$ is Cauchy $\iff$ $\{X_n\}_n$ is mutually convergent sequence

PROP 2 $[X_n \longrightarrow X] = \left\{\omega \in \Omega \mid X_n(\omega) \xrightarrow[n\to\infty]{} X(\omega)\right\}$ is the convergent se…

$$= \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} \left[\,|X_m - X| < \frac{1}{k}\right]$$

Note $[X_n \longrightarrow X]^c = N$

$$[X_n \longrightarrow X] = \left\{\omega \in \Omega \;\middle|\; X_n(\omega) \xrightarrow[n\to\infty]{} X(\omega)\right\}$$

Now $X_n(\omega) \longrightarrow X(\omega) \iff \forall\, \varepsilon > 0 \;\; \exists\, n_\varepsilon \geq 1$ st

$$|X_n(\omega) - X(\omega)| \leq \varepsilon \qquad \forall\, n \geq n_\varepsilon$$

ie $\iff$ $\varepsilon = \dfrac{1}{k}$ $\forall\, k > 0$ $\exists\, n_k \geq 1$ st

$$|X_n(\omega) - X(\omega)| \leq \frac{1}{k} \qquad \forall\, n \geq n_k$$

$$X_n(\omega) \longrightarrow X(\omega) \iff \forall\ 1/k \quad \exists\ n_k \geq 1 \quad \text{st} \quad \bigcap_{n=n_k}^{\infty} \left(|X_n(\omega) - X(\omega)| \leq 1/k \right)$$

$$\iff \bigcap_{k=1}^{\infty} \ \bigcup_{n_k=1}^{\infty} \ \bigcap_{n=n_k}^{\infty} \left[|X_n(\omega) - X(\omega)| \leq 1/k \right]$$

$$\iff \underbrace{\bigcap_{k=1}^{\infty} \ \bigcup_{n=1}^{\infty} \ \bigcap_{m=n}^{\infty}}_{\text{change of letters for the index}} \left(|X_m(\omega) - X(\omega)| \leq 1/k \right)$$

change of letters for the index

<u>The divergent set</u> $[X_n \to X]^c = \underline{\bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} \left(|X_m - X| > \frac{1}{k} \right)}$

$$\underline{X_n \xrightarrow[n\to\infty]{a.e.} X \iff \mu\left([X_n \to X]^c\right) = 0}$$

$$X_n \xrightarrow[n\to\infty]{a.e} X \iff X_n(\omega) \longrightarrow X(\omega) \quad \forall\ \omega \in N^c,\ \mu(N) = 0$$

$$\iff \mu\left(\omega \in \Omega \,\middle|\, X_n(\omega) \xrightarrow[n\to\infty]{} X(\omega)\right) = 1$$

$$X_n \xrightarrow[n\to\infty]{a.e} X \iff X_m - X_n \xrightarrow[n\to\infty]{a.e} 0$$

We can say: $X_n \xrightarrow[n\to\infty]{a.e} X \iff \forall\ \varepsilon > 0 \ \exists\ n_\varepsilon$ st $|X_m(\omega) - X_n(\omega)| \leq \varepsilon$

$\forall\ m, n \geq n_\varepsilon$ & $\forall\ \omega \in N^c$ with

$\mu(N) = 0$

Fix $\varepsilon > 0$ arbitrary

$$\Leftrightarrow \quad \mu\left(\bigcup_{n=1}^{\infty} \bigcap_{m\geq n}^{\infty} \left(\omega \,\middle|\, |X_m(\omega) - X_n(\omega)| \leq \varepsilon\right)\right) = 1 \qquad \forall\, \varepsilon > 0$$

$$\Leftrightarrow \quad \mu\left(\bigcap_{n=1}^{\infty} \bigcup_{m\geq n}^{\infty} \left(\omega \,\middle|\, |X_m(\omega) - X_n(\omega)| > \varepsilon\right)\right) = 0$$

$$\Leftrightarrow \quad \mu\left(\overline{\lim}\, \left[|X_m - X_n| > \varepsilon\right]\right) = 0 \qquad \forall\, \varepsilon > 0.$$

$$\Leftrightarrow \quad \mu\left(\overline{\lim}\, \left(|X_n - X| > \varepsilon\right)\right) = 0 \qquad \forall\, \varepsilon > 0.$$

Now let $\quad A_n = \bigcup_{m\geq n}^{\infty} \left[|X_m - X_n| > \varepsilon\right]$

then $\quad \{A_n\}_n$ is an $\downarrow$ seq

$$\therefore \quad \mu\left(\bigcap_{n=1}^{\infty} \bigcup_{m\geq n}^{\infty} \left[|X_m - X_n| > \varepsilon\right]\right) = 0 \qquad \forall\, \varepsilon > 0.$$

$$\underset{\text{if } \mu(\Omega)<\infty}{\overset{\uparrow}{\Leftrightarrow}} \quad \mu\left[\bigcap_{n=1}^{\infty} A_n\right] = \lim_{n\to\infty} \mu(A_n) = 0 \qquad \forall\, \varepsilon >$$

$$\text{ie} \quad \mu\left[\bigcap \bigcup \left[|X_m - X_n| > \varepsilon\right]\right] = \lim_{n\to\infty} \mu\left[\bigcup_{m\geq n}^{\infty} \left[|X_m - X_n| > \varepsilon\right]\right]$$

$$\therefore \quad X_n \xrightarrow{a.e} X \quad \Leftrightarrow \quad \forall\, \varepsilon > 0 \quad \lim_{n\to\infty} \mu\left(\bigcup_{m\geq n}^{\infty} \left[|X_n - X_m| > \varepsilon\right]\right) = $$

$$\Leftrightarrow \forall\, \varepsilon > 0 \ \exists\, n_\varepsilon \geq 1 \text{ st } \forall\, n \geq n_\varepsilon \text{ we have}$$

$$\mu\left[\bigcup_{m\geq n}^{\infty} \left[|X_n - X_m| > \varepsilon\right]\right] \leq \varepsilon$$

$X_n \xrightarrow{a.e} X \iff \forall\, \varepsilon>0,\ \exists\, n_\varepsilon \geq 1$ st $\forall\ m \geq n \geq n_\varepsilon$ we have

$$\mu\left(\bigcup_{m\geq n_\varepsilon}^{\infty}\left[|X_m - X_n| > \varepsilon\right]\right) \leq \varepsilon$$

$$\iff \forall\, \varepsilon>0\quad \exists\, n_\varepsilon>0 \quad \text{st} \quad \mu\left[\bigcup_{N=n_\varepsilon}^{\infty}\ \bigcup_{m\geq n_\varepsilon}^{N}\left[|X_m - X_n|>\varepsilon\right]\right] \leq \varepsilon$$

Let $A_N = \bigcup_{m\geq n_\varepsilon}^{N}\left[|X_m - X_n| > \varepsilon\right]$

then $A_{n_\varepsilon} = \left[|X_{n_\varepsilon} - X_n| > \varepsilon\right]$

$A_{n_\varepsilon+1} = \left[|X_{n_\varepsilon} - X_n| > \varepsilon\right] \cup \left[|X_{n_\varepsilon+1} - X_n| > \varepsilon\right]$

$\{A_N\}_N$ is an $\uparrow$ seq

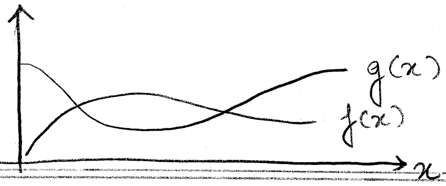
$$X_n \xrightarrow[n\to\infty]{a.e} X \iff \forall\, \varepsilon>0,\ \exists\, n_\varepsilon\geq 1 \ \text{st}\ \mu\left(\bigcup_{N=n_\varepsilon}^{\infty} A_N\right) \leq \varepsilon \quad \forall\ n\geq n_\varepsilon$$

$$\iff \forall\, \varepsilon>0,\ \exists\, n_\varepsilon\geq 1 \ \text{st}\ \lim_{N\to\infty}\mu(A_N) \leq \varepsilon \quad \forall\ n\geq n_\varepsilon$$

$$\iff \forall\, \varepsilon>0,\ \exists\, n_\varepsilon\geq 1 \ \text{st}\ \lim_{N\to\infty}\mu\left[\bigcup_{m\geq n_\varepsilon}^{N}\left(|X_m - X_n|>\varepsilon\right)\right] \leq \varepsilon$$

$$\iff \forall\, \varepsilon>0,\ \exists\, n_\varepsilon\geq 1 \ \text{st}\ \forall\ N\geq n\geq n_\varepsilon \ \text{we have}\ \mu\left[\bigcup_{m=n_\varepsilon}^{N}|X_m - X_n|>\varepsilon\right] \leq \varepsilon$$

Note $\quad \bigcup_{j\in J_0}\left(|y_j|>a\right) = \max_{1\leq j\leq J_0} |y_j| > a$



max $\{f(x), g(x)\}$ does not have to be $f(x)$ or $g(x)$

$$\bigcup_{j=1}^{J_0} \{\omega \in \Omega \mid |Y_j(\omega)| > a\} = \{\omega \in \Omega \mid (\max_{1\le j\le J_0} |Y_j|)(\omega) >$$

$$\iff \bigcap_{j \in J_0} [|Y_j| \le a] = \{\max_{1\le j\le J_0} |Y_j| \le a\}$$

Proof $\quad \omega \in \bigcap_{j\in J_0} |Y_j| \le a \rightarrow \omega \in |Y_j| \le a \quad \forall j$

$$\rightarrow \omega \in \max_{1\le j\le J_0} |Y_j| \le a$$

$$\therefore \bigcap_{j\in J_0} |Y_j| \le a \subseteq \{\max_{1\le j\le J_0} |Y_j| \le a\}$$

$$\therefore \boxed{X_n \xrightarrow[n\to\infty]{a.e} X \iff \forall \varepsilon>0 \ \exists n_\varepsilon \ge 1 \text{ st } \forall N \ge n \ge n_\varepsilon \text{ we have } \mu\left(\max_{n\le m\le N} |X_m - X_n| > \varepsilon\right) \le \varepsilon}$$

HWK Due Oct 25^{th}

1.) Ex 2.3.3 Pg 32

2) Ex 2.4.1 Pg 34

3) Ex 2.4.2 Pg 34

$P(|u+v| \geq 2\varepsilon) \leq P(|u| \geq \varepsilon) + P(|v| \geq \varepsilon)$

CONVERGENCE IN MEASURE

Let $\{X_n\}_n$ be a seq of measurable functions

$$\boxed{X_n \xrightarrow[n\to\infty]{\mu} X \iff \forall\, \varepsilon > 0 \quad \mu\left(\{\omega \in \Omega \mid |X_n(\omega) - X(\omega)| > \varepsilon\}\right) \xrightarrow[n\to\infty]{} 0}$$

1. Pointwise Convergence $X_n(\omega) \longrightarrow X(\omega)\ \forall\, \omega \in \Omega$
2. Almost Everywhere conv $X_n \xrightarrow[n\to\infty]{a.e} X \iff X_n(\omega) \longrightarrow X(\omega)\ \forall\, \omega \in N^c$ where $\mu(N) = 0$
(stronger than conv in meas)

The limit in measure is almost everywhere unique
ie If $X_n \xrightarrow[n\to\infty]{\mu} X$ & $X_n \xrightarrow[n\to\infty]{\mu} \tilde{X}$ then $X = \tilde{X}$ a.e

Pf

$\mu(X \neq \tilde{X}) = 0$ is what we need to show

$$0 \leq \mu(X \neq \tilde{X}) = \mu\left(\bigcup_{k=1}^{\infty} |X - \tilde{X}| > 1/k\right) \qquad \left\{ \text{since } X \neq \tilde{X} \Leftrightarrow |X - \tilde{X}| > 0 \right.$$

$$\leq \sum_{k=1}^{\infty} \mu\left(|X - \tilde{X}| > 1/k\right)$$

We will show: $\forall\, \varepsilon > 0$, $\mu(|X - \tilde{X}| > 2\varepsilon) = 0$

$$0 \leq \mu(X \neq \tilde{X}) \leq \mu\left[\left(|X - \tilde{X}| > 2\varepsilon\right)\right] = \lim_{n\to\infty} \mu\left(|X - \tilde{X}| \geq 2\varepsilon\right)$$

$$\therefore 0 \leq \mu(X \neq \tilde{X}) \leq \lim_{n\to\infty} \mu\left[\left(|\underbrace{X - X_n}_{u} + \underbrace{X_n - \tilde{X}}_{v}| \geq 2\varepsilon\right)\right]$$

$$0 \leq \mu(X \neq \tilde{X}) \leq \lim_{n\to\infty} \mu\left(|X_n - X| \geq \varepsilon\right) + \lim_{n\to\infty} \mu\left(|X_n - \tilde{X}| \geq \varepsilon\right)$$

$$\therefore 0 \leq \mu(X \neq \tilde{X}) \leq 0 \implies \mu(X \neq \tilde{X}) = 0$$

Property $\boxed{P(|u+v| \geq 2\varepsilon) \leq P(|u| \geq \varepsilon) + P(|v| \geq \varepsilon)}$

(*) $|u+v| \geq 2\varepsilon \implies |u| \geq \varepsilon$ OR $|v| \geq \varepsilon$

(*) implies $P(|u+v| \geq 2\varepsilon) \leq P(|u| \geq \varepsilon \cup |v| \geq \varepsilon)$

$\leq P(|u| \geq \varepsilon) + P(|v| \geq \varepsilon)$

Now if $p \rightarrow q$ then $\sim q \rightarrow \sim p$

$|u| \leq \varepsilon$ and $|v| \leq \varepsilon \implies |u+v| \leq 2\varepsilon$

$\rightarrow |u+v| \leq |u| + |v|$

$\therefore \sim q \rightarrow \sim p$ and so (*) holds

PROP $\underline{X_n \xrightarrow{\mu} X \;\not\rightarrow\; X_n \xrightarrow{a.e} X}$

THEOREM The link b/w a.e convg and convg in meas

1) $X_n \xrightarrow[n\to\infty]{a.e} X \iff X_m - X_n \xrightarrow[m \wedge n \to \infty]{a.e} 0$

2) $X_n \xrightarrow[n\to\infty]{\mu} X \iff X_m - X_n \xrightarrow[m \wedge n \to \infty]{\mu} 0$

3) If $\underline{\mu(\Omega) < \infty}$ then $X_n \xrightarrow[n\to\infty]{a.e} X \implies X_n \xrightarrow[n\to\infty]{\mu} X$

Proof We need to show $\forall \varepsilon > 0$, $\mu(|X_n - X| > \varepsilon) \xrightarrow[n\to\infty]{} 0$

If we can bdd above the seq of nos (**) by another seq which $\to 0$ then our seq also $\to 0$

one memb of the union

$$0 \leq \mu\left[|X_n - X| > \varepsilon\right] \leq \mu\left(\bigcup_{m \geq n}^{\infty}\left[|X_m - X| > \varepsilon\right]\right)$$

Now If $\mu(\Omega) < \infty$ then $X_n \xrightarrow[n\to\infty]{a.e} X \iff \mu\left(\bigcup_{m\geq n}^{\infty}\left[|X_m - X| \geq \varepsilon\right]\right) \xrightarrow[n\to\infty]{} 0$

$$\therefore \quad 0 \leq \mu\left(|X_n - X| > \varepsilon\right) \leq \mu\left(\bigcup_{m\geq n}^{\infty} |X_m - X| \geq \varepsilon\right) \longrightarrow 0$$

1) If $\Omega = [0,\infty)$ & $\mu = \lambda$ then $\mu(\Omega) = \infty$ and the a.e conv $\not\to$ always conv in measure

2) Convergence in measure $\not\Rightarrow$ conv almost everywhere

3) If μ is a probability measure then you use a.s inplace of a.e

$\boxed{\text{EM}}$ $X_n \xrightarrow{\mu} X \implies \exists \{X_{n_k}\}_k$ a subsequence of $\{X_n\}_n$ st $X_{n_k} \xrightarrow[k\to\infty]{a.e} X$

DEF Given 2 measurable spaces $(\Omega, \mathcal{A})$ and $(\tilde{\Omega}, \tilde{\mathcal{A}})$
X is $\tilde{\mathcal{A}}-\mathcal{A}$ measurable if $X^{-1}(\tilde{\mathcal{A}}) \subset \mathcal{A}$

Define $\mathcal{F}(X) = \{ X^{-1}(A') : A' \in \tilde{\mathcal{A}} \} \subset \mathcal{A}$
$\mathcal{F}(X)$ is called a sub σ-field induced by the function X

THEOREM Let $Y : (\Omega, \mathcal{F}(X)) \longrightarrow (\overset{\mathbb{R}}{\tilde{\Omega}}, \overset{\mathcal{B}}{\tilde{\mathcal{A}}})$ and Y is $\tilde{\mathcal{A}} - \mathcal{F}(X)$ meas
and $X : (\Omega, \mathcal{A}, \mu) \longrightarrow (\mathbb{R}, \mathcal{B})$ which is $\mathcal{B}$-$\mathcal{A}$ measurable
If Y is $\tilde{\mathcal{A}} - \mathcal{F}(X)$ measurable then $\exists$ a borel measurable funct
$g(\cdot)$ $[\, g : (\mathbb{R}, \mathcal{B}) \longrightarrow (\mathbb{R}, \mathcal{B}) \,]$ such that $Y = g \circ X$

INDUCED MEASURE

Given a measure space $(\Omega, \mathcal{A}, \mu)$ and a measurable
$(\tilde{\Omega}, \tilde{\mathcal{A}})$
Given $X : \Omega \longrightarrow \tilde{\Omega}$ is an $\tilde{\mathcal{A}} - \mathcal{A}$ measurable function

We define a measure $\tilde{\mu}$ on $(\tilde{\Omega}, \tilde{\mathcal{A}})$ as $\boxed{\tilde{\mu}(A') = \mu(X^{-1}(A')) =}$

To check $\mu_X(\cdot)$ is a measure on $(\tilde{\Omega}, \tilde{\mathcal{A}})$

(i) $\mu_X(A') = \mu(X^{-1}(A)) = \mu(B) \geq 0$ $\Big\{$ since $\mu(\cdot)$ is a meas
$X^{-1}(A) = B$ for some B

(ii) $\mu_X(\phi) = \mu(X^{-1}(\phi)) = \mu(\phi) = 0$

(iii) $\mu_X\left(\sum_{n=1}^{\infty} A'_n\right) = \mu\left(X^{-1}\left(\sum_{n=1}^{\infty} A'_n\right)\right)$ $\Big\{$ A_n's are disjoint subsets
$A_n \in \tilde{\mathcal{A}}$

$$= \mu\left(\sum_{n=1}^{\infty} X^{-1}(A'_n)\right)$$

Now if A'_n are disjoint, $X^{-1}(A_n)$ are also disjoint
Also $X^{-1}(A'_n) \in \mathcal{A}$ $\forall\, n \geq 1$ since μ is $\tilde{\mathcal{A}}$-$\mathcal{A}$ measurable

$$\therefore \quad \mu\left(\sum_{n=1}^{\infty} X^{-1}(A_n)\right) = \sum_{n=1}^{\infty} \mu\left(X^{-1}(A'_n)\right) = \sum_{n=1}^{\infty} \mu_X(A'_n)$$

$$\therefore \quad \mu_X\left(\sum_{n=1}^{\infty} A'_n\right) = \sum_{n=1}^{\infty} \mu(A'_n)$$

NOTE: If $\mu(\cdot)$ is a probability measure then the induced measure $\mu_X(\cdot)$ is also a probability measure

Proof

$$\mu_X(\tilde{\Omega}) = \mu\left(X^{-1}(\tilde{\Omega})\right) = \mu(\Omega) = 1$$

QED

RELATION B/W INDUCED MEASURE & DISTRIBUTION FUNCT$^{\text{N}}$

Every induced meas can be uniquely characterized by df.
$X: (\Omega, \mathcal{A}, \mu) \longrightarrow (\tilde{\Omega}, \tilde{\mathcal{A}})$ where μ is a prob measure $= P$

$$P(\{\omega : X(\omega) \in B\}) = \underbrace{P(X^{-1}(B))}_{\text{is also a prob measure.}} \quad \& \quad B \in \tilde{\mathcal{A}}$$

Now if $X : (\Omega, \mathcal{A}, P(\cdot)) \longrightarrow (\mathbb{R}, \mathcal{B})$
ie if $\tilde{\Omega} = \mathbb{R}$

P is a prob measure $\Rightarrow$ $P(X^{-1}(\cdot))$ is also a prob meas
& since $\tilde{\Omega} = \mathbb{R}$ & $\tilde{\mathcal{A}} = \mathcal{B}$ it is a L-S meas

From the corollary of $\underbrace{\text{correspondence theorem}}_{\text{can be used only for LS meas \& this holds only since } \tilde{\Omega} = \mathbb{R}}$ we can say

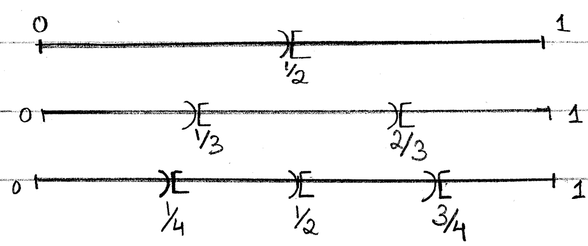
COUNTEREXAMPLES

1) $X_n \xrightarrow[n\to\infty]{\mu} X \quad \not\Longrightarrow \quad X_n \xrightarrow{a.e} X$

We will study a seq $\{X_n\}_n$ st $X_n \xrightarrow{\mu} X$ but $X_n \overset{a.e}{\not\to} X$

$X_n : (\Omega, \mathcal{A}, \mu) \longrightarrow (\mathbb{R}, \mathcal{B})$ where $\Omega = [0,1)$, $\mathcal{A} = \mathcal{B}$

$\mu = \lambda$ the lebesgue m



Define $X_1 = I_{[0,1)}$

$X_2 = I_{[0,\frac{1}{2})}$

$X_3 = I_{[\frac{1}{2},1)}$

$X_4 = I_{[0,\frac{1}{3})}$

ie $X_{\frac{m(m-1)}{2}+k} = I_{[\frac{k-1}{m},\frac{k}{m})}$ where $m \geq 1$ & $k = 1,2 \ldots m$

Define $X = 0$

To show $\forall \varepsilon > 0$ $\mu(\{\omega : |X_n(\omega) - X(\omega)| > \varepsilon\}) \longrightarrow 0$ as $n \to \infty$

ie $\forall \varepsilon > 0, \delta > 0$, $\exists$ N st $\forall n \geq N$; $\mu\left(\{\omega \in \Omega \mid |X_n(\omega) - X(\omega)| > \varepsilon\}\right) < \delta$

Now $\mu(\{\omega : |X_n(\omega) - X(\omega)| > \varepsilon\}) = \mu(\{\omega : |X_n(\omega)| > \varepsilon\})$

Since $X_n(\omega)$'s are all indicator functions $\forall n \geq 1$ the

$\{\omega \in \Omega : |X_n(\omega)| > \varepsilon\} = \{\omega \in \Omega : X_n(\omega) = 1\} = A_n$

where $A_{\frac{m(m-1)}{2}+k} = \left[\frac{k-1}{m}, \frac{k}{m}\right)$

$\mu\left(\{\omega \in \Omega : |X_n(\omega)| > \varepsilon\}\right) = \mu(A_n) \longrightarrow 0$ as $n \to \infty$

since lengths of $A_n \downarrow 0$ as $n \to \infty$

$\therefore \quad X_n \xrightarrow{\mu} X = 0$

Now we will show $X_n \xrightarrow[]{a.e} \!\!\!\!\!\!/ \;\; X$ as $n \to \infty$

Now we will show $\lim X_n(\omega)$ does not exist $\forall\, \omega \in \Omega$

and hence it cannot converge to anything

We will show $\overline{\lim}\, X_n(\omega) \neq \underline{\lim}\, X_n(\omega) \quad \forall\, \omega \in \Omega$ {not ju $\omega \in$

$$\text{Now} \quad \overline{\lim}\, X_n(\omega) = \lim_{n\to\infty} \sup_{m \geq n} X_m(\omega)$$

$$= \lim_{n\to\infty} (1)$$

$$= 1$$

$\sup_{m\geq n} X_m(\omega) =$

Given any N $\exists A_n$ st $\omega \in A_n \Rightarrow$ I

$$\underline{\lim}\, X_n(\omega) = \lim_{n\to\infty} \inf_{m \geq n} X_m(\omega)$$

$$= \lim_{n\to\infty} 0$$

$$= 0$$

{Given any N, $\exists$ at least one inte to which $\omega \notin$

$\therefore \quad \overline{\lim}\, X_n(\omega) \neq \underline{\lim}\, X_n(\omega) \Longrightarrow$ limit does not exist $\forall$ ω

$\therefore \quad X_n(\omega) \xrightarrow[a.e]{} \!\!\!\!\!\!/ \;\; X(\omega)$

So $\quad X_n \xrightarrow{\mu} X \quad \not\Rightarrow \quad X_n \xrightarrow{a.e} X$

If $\mu(\Omega) < \infty$ then $X_n \xrightarrow{a.e} X \Rightarrow X_n \xrightarrow{\mu} X$

If $\mu(\Omega) = \infty$ then $X_n \xrightarrow{a.e} X \not\Rightarrow X_n \xrightarrow{\mu} X$

Let $\Omega = [0,\infty)$ $\mathcal{A} = \mathcal{B}_{[0,\infty)}$ $\mu = \lambda$ the lebesgue measure

Clearly $\mu(\Omega) = \infty$

Define $X_n = I_{[n,n+1)}$ $n = 0,1,2\dots$

Define $X(\omega) = 0$ $\forall\ \omega \in \Omega$

We will show $X_n \xrightarrow{a.e} X$ ie $X_n(\omega_0) \longrightarrow X(\omega_0)$ for arbitrary fixed $\omega_0 \in \Omega$

For fixed $\omega_0 \in [0,\infty)$ $\exists\ n_0$ st $\omega_0 \in [n_0, n_0+1)$

For this n_0, $X_{n_0}(\omega_0) = 1$ and $\forall\ n \neq n_0$ $X_n(\omega_0) = 0$

$\therefore \lim_{n\to\infty} X_n(\omega_0) = 0 = X(\omega_0)$

$\therefore X_n(\omega) \xrightarrow{a.e} X(\omega)$

Now we will show $\mu\{\omega : |X_n(\omega) - X(\omega)| > \varepsilon\} \not\to 0$ for some $\varepsilon > 0$

$$\begin{aligned}
\mu(\{\omega : |X_n(\omega) - X(\omega)| > \varepsilon\}) &= \mu(\{\omega : |X_n(\omega)| = 1\}) \\
&= \mu(\{\omega : \omega \in [n,n+1)\}) \\
&= \lambda([n,n+1)) \\
&= 1 \quad \not\to 0
\end{aligned}$$

$\therefore$ For any $\varepsilon > 0$ $\mu(\{\omega : |X_n(\omega) - X(\omega)| > \varepsilon\}) \not\to 0$ as $n \to \infty$

★ $P(X^{-1}(a,b]) = F(b) - F(a)$ where $F(\cdot)$ is a $dist^n$ funct st $F(\infty) = 1$ & $F(-\infty) =$

NOTE ① It is usually easier to work with $F(\cdot)$ then $P(X^{-1}(\cdot))$

② $P(X^{-1}(\cdot))$ is defined on $\mathcal{B}$ & so makes sense when applied to intv of type $(a,b]$

$P(\cdot)$ is defined on $\mathcal{A}$ which is not Borel & makes no sense when applied to $(a,b]$

DEF $X_n \xrightarrow{d} X$

A seq $\{X_n\}_{n \geq 1}$ is said to converge in law OR distribution to a rv X

$F_{X_n}(x) \longrightarrow F_X(x) \quad \forall \; x \in$ continuity pts of $F(\cdot)$

NOTE: The real definition is $P(X_n^{-1}(A)) \longrightarrow P(X^{-1}(A))$ for $A \in$

$P(X^{-1}(\partial A)) = 0$

which is complicated and this is where the rel^n b/w indu meas & $dist^n$ functions helps to simplify things

THEOREM $X_n \xrightarrow{P} X \implies X_n \xrightarrow{d} X$ for $X =$ const

[If X is not constant the implication does not hold.

Proof To show $X_n \xrightarrow{P} X \implies \lim_{n\to\infty} F_n(x) = F(x) \quad \forall \; x \in C(F)$ continuity pts of

ie $X_n \xrightarrow{P} X \implies \forall \varepsilon > 0 \; \exists \; N_\varepsilon$ st $|F_n(x) - F(x)| \leq \varepsilon \quad \forall \; n \geq N_\varepsilon$

$\forall \; x \in C(F)$

ie to show $-\varepsilon \leq F_n(x) - F(x) \leq \varepsilon$ for $n \geq N_\varepsilon$

ie $F(x) - \varepsilon \leq F_n(x)$ & $F_n(x) \leq F(x) + \varepsilon$

To show $F_{X_n}(x) \leq F_X(x) + \varepsilon$

$$F_{X_n}(x) = P(X_n \leq x) = P\left((X_n + X - X) \leq x \right)$$

For $\varepsilon' > 0$ arbitrary

If $X \geq x + \varepsilon'$ & $X_n - X \geq -\varepsilon' \Rightarrow X_n \geq x$

ie $\{X \geq x + \varepsilon'\} \cap \{X_n - X \geq -\varepsilon'\} \subseteq \{X_n \geq x\}$

DeMorgans Law

$\therefore \{X \leq x + \varepsilon'\} \cup \{X_n - X \leq -\varepsilon'\} \supseteq \{X_n \leq x\}$

$$\begin{aligned}\therefore F_{X_n}(x) &= P(X_n + X - X \leq x) \\ &\leq P(\{X_n - X \leq -\varepsilon'\} \cup \{X \leq x + \varepsilon'\}) \\ &\leq P\{X \leq x + \varepsilon'\} + P\{X_n - X \leq -\varepsilon'\}\end{aligned}$$

$$\{|X_n - X| \geq \varepsilon'\} = \{X_n - X \geq \varepsilon'\} \cup \{X_n - X \leq -\varepsilon'\}$$

$\therefore P\{|X_n - X| \geq \varepsilon'\} \geq P\{X_n - X \leq -\varepsilon'\}$

$\therefore F_{X_n}(x) \leq P\{X \leq x + \varepsilon'\} + P\{|X_n - X| \geq \varepsilon'\}$

$$F_{X_n}(x) \leq F_X(x + \varepsilon') + P(|X_n - X| \geq \varepsilon')$$

$$F_{X_n}(x) \leq F_X(x + \varepsilon') + P(|X_n - X| \geq \varepsilon')$$

$$F_{X_n}(x) \leq F_X(x + \varepsilon') + \varepsilon \qquad \forall\, n \geq n_1$$

$\left\{ \text{since } X_n \xrightarrow{P} X \Rightarrow P(|X_n - X| \geq \varepsilon') \leq \varepsilon \quad \forall\, n \geq n_1 \right\}$

$$\lim_{\varepsilon' \to 0} F_{X_n}(x) \leq \lim_{\varepsilon' \to 0} F_X(x + \varepsilon') + \lim_{\varepsilon' \to 0} \varepsilon$$

$\therefore F_{X_n}(x) \leq F_X(x) + \varepsilon \quad \forall\, n \geq n_1$

$\left\{ \text{by rt continuity } \lim_{\varepsilon' \to 0} F_X(x + \varepsilon') = F_X(x) \right.$

HW 4

$X_n \xrightarrow{P} c \iff X_n \xrightarrow{d} c$

$$P(AB) = P(A) - P(AB^c) \geq P(A) - P(B^c)$$

To show $F_{X_n}(x) \geq F_X(x) - \varepsilon$ for some $n \geq n_2$

Now $F_{X_n}(x) = P(X_n \leq x)$

$$\{X \leq x - \varepsilon'\} \cap \{|X_n - X| \leq \varepsilon'\} \subseteq \{X_n \leq x\}$$

$$P(X_n \leq x) \geq P((X \leq x\ \varepsilon') \cap (|X_n - X| \leq \varepsilon'))$$

$$\therefore F_{X_n}(x) \geq P(\{X \leq x - \varepsilon'\} \cap \{|X_n - X| \leq \varepsilon'\})$$

$$\geq P(\{X \leq x - \varepsilon'\}) - P(|X_n - X| \geq \varepsilon')$$

Since $X_n \xrightarrow{P} X \implies \exists\, n_2$ st $P(|X_n - X| \geq \varepsilon') \leq \varepsilon \quad \forall\ n \geq n_2$

$\implies -P(|X_n - X| \geq \varepsilon') \geq -\varepsilon$

$$F_{X_n}(x) \geq F_X(x - \varepsilon') - \varepsilon \qquad \forall\ n \geq n_2$$

Now if x is a continuity pt of F then $\lim_{\varepsilon' \to 0} F_X(x - \varepsilon') = F_X(x)$

at other pts this need not hold since $F_X(\cdot)$ is only rt con

$$\therefore \lim_{\varepsilon' \to 0} F_{X_n}(x) \geq \lim_{\varepsilon' \to 0} F_X(x - \varepsilon') - \varepsilon$$

$$F_{X_n}(x) \geq F_X(x) - \varepsilon \qquad \forall\ n \geq n_2 \ \ \&\ x \in C(F)$$

let $N = \max\{n_1, n_2\}$ then $\forall\ n \geq N$ we have

$$F_{X_n}(x) \geq F_X(x) - \varepsilon \quad \text{and} \quad F_{X_n}(x) \leq F_X(x) + \varepsilon$$

$$\therefore F_{X_n}(x) \longrightarrow F_X(x) \quad \forall\ x \in C(F)$$

ie $X_n \xrightarrow{d} X$

1 Slutsky's Theorem : If $X_n \xrightarrow{d} X$, $Y_n \xrightarrow{P} a$, $Z_n \xrightarrow{P} b$
$\Rightarrow Y_n \cdot X_n + Z_n \xrightarrow{d} aX + b$

: Find on your own

EM (Done before)

$X : (\Omega, \mathcal{A}) \longrightarrow (\tilde{\Omega}, \tilde{\mathcal{A}})$ is $\tilde{\mathcal{A}}$-$\mathcal{A}$ measurable

Now $X^{-1}(\tilde{\mathcal{A}}) \subset \mathcal{A}$
$X^{-1}(\tilde{\mathcal{A}})$ is a σ-field
$X^{-1}(\tilde{\mathcal{A}}) \equiv \mathcal{F}(X)$
$Y : (\Omega, \mathcal{F}(X)) \longrightarrow (\mathbb{R}, \mathcal{B})$

EM If $(\tilde{\Omega}, \tilde{\mathcal{A}}) \equiv (\mathbb{R}, \mathcal{B})$ and given a variable Y which is $\mathcal{B}$-$\mathcal{F}(X)$ measurable then $\exists\ g : (\mathbb{R}, \mathcal{B}) \longrightarrow (\mathbb{R}, \mathcal{B})$ st $Y = g(X)$

(This theorem does not tell us what g is however)

Cases

<1> Let $Y \geq 0$: Y is a simple funct^n

<2> $Y = Y^+ - Y^-$ where $Y^+ = Y \cdot I\{\omega : Y(\omega) \geq 0\}$, $Y^- = -Y \cdot I\{\omega : Y(\omega) \leq 0\}$ both ≥ 0
(Y: any funct^n)

Y^+ & Y^- are both measurable

<3> If $X(\omega) \geq 0$, $\exists$ $\{X_n(\omega)\} \uparrow$ simple functions st $\lim_{n \to \infty} X_n(\omega) = X(\omega)$

Case : $\langle 1 \rangle$: Y is a simple funct^n

If $Y = I_A$ where $\underline{A \in \mathcal{F}(X)}$ $\leftarrow$ since Y is $\mathcal{B}-\mathcal{F}(X)$ measura

$\therefore$ $Y = I_{\{\omega : X(\omega) \in B\}}$ where $B \in \mathcal{B}$ st $A = X^{-1}(B)$

$= I_{\{X(\omega) \in B\}} (X(\omega))$

$= g(X)$ where $g(X) = I_B$

If $Y = \sum_{i=1}^{n} a_i I_{A_i}$ where $A_i \in \mathcal{F}(X)$

$Y = g(X)$ where $g(X) = \sum_{i=1}^{n} a_i I_{B_i}$; B_i are st $A_i =$

$\langle 2 \rangle$ Let $Y \geq 0$ arbitrary funct^n

$Y = \lim_{n \to \infty} Y_n$ where Y_n are simple funct^n (by theo

$Y = \lim_{n \to \infty} g_n(X)$ $\{$ since by $\langle 1 \rangle$ all simple funct^n $Y = g$

We know g_n's are measurable $\Rightarrow \lim_{n \to \infty} g_n = g$ is also measu

$\therefore$ $Y = g(X)$ is measurable.

Now $Y = Y^+ - Y^-$

$= g^+(X) - g^-(X)$ $\left\{ \text{since } Y^+, Y^- \text{ are } \geq 0 \text{ \& measurable} \right.$

$= g(X)$

INTEGRALS

DEF The Lebesgue Integral $\int X$ OR $\int X \, d\mu$ denoted by $E(X)$

1) If $X = \sum_{i=1}^{n} x_i I_{A_i}$ st $\sum_{i=1}^{n} A_i = \Omega$ & $x_i \geq 0$

then $\int X \, d\mu \equiv \sum_{i=1}^{n} x_i \, \mu(A_i)$

2) If $X \geq 0$ [a positive functⁿ], measurable then

$$\int X \, d\mu = \sup \left\{ \int Y \, d\mu \; ; \; 0 \leq Y \leq X \text{ and } Y \text{ is a simple} \right.$$

3) If X is any general measurable functⁿ
we can write $X = X_+ - X_-$ where $X_+ = \max(X, 0)$
$X_- = \max(-X, 0)$

then

$$\int X \, d\mu \equiv \int X^+ \, d\mu - \int X^- \, d\mu$$

NOTE 1) The integral defined as above then the integral is unique
2) If X is non-measurable but $X = Y$ a.e and Y is measurable then $\int X = \int Y$

DEF If $\int X \, d\mu < \infty$ then X is called an integrable funct

Ques: Does the Lebesgue integral have the same properties as a Reimann integral

2 a) $\int (X+Y)\, d\mu = \int X\, d\mu + \int Y\, d\mu$

b) $\int cX\, d\mu = c \int X\, d\mu$ for some c

c) $X \geq 0 \implies \int X\, d\mu \geq 0$ [this prop implies if $X \geq Y \to \int X d\mu \geq \int Y d\mu$]

a) Step 1 Let $X = \sum_i^m x_i I_{A_i}$ & $Y = \sum_j^n y_j I_{B_j}$ (simple funct^{n})

$$X + Y = \sum_i^m \sum_j^n (x_i + y_j)\, I_{A_i B_j}$$

$$\int (X+Y)\, d\mu = \sum_i^m \sum_j^n (x_i + y_j)\, \mu(A_i B_j)$$

$$= \sum_i^m \sum_j^n (x_i)\, \mu(A_i B_j) + \sum_i^m \sum_j^n (y_j)\, \mu(A_i B_j)$$

$$= \sum_{i=1}^m x_i \sum_{j=1}^n \mu(A_i B_j) + \sum_{j=1}^n y_j \sum_{i=1}^m \mu(A_i B_j)$$

$$= \sum_{i=1}^m x_i \left[\mu\left(\sum_{j=1}^n A_i B_j \right) \right] + \sum_{j=1}^n y_j\, \mu\left(\sum_{i=1}^m A_i B_j \right)$$

$$= \sum_{i=1}^m x_i\, \mu(A_i) + \sum_j^n y_j\, \mu(B_j)$$

A_i & B_j form partitions of Ω

$$= \int X d\mu + \int Y\, d\mu$$

The Monotone Convergence Theorem (MCT)

Let $\{X_n\}_n$ be a seq of $\uparrow$ measurable functions st $X_n \geq 0$ and $X_n \uparrow X$ a.e then $\int X_n \uparrow \int X$

$$\left[\lim_{n\to\infty} X_n = X \text{ a.e} \implies \lim_{n\to\infty} \int X_n = \int \lim_{n\to\infty} X = \int X \right]$$

$$\left\{\begin{array}{ll} X_{n+1} \geq X_n \geq X_{n-1} \cdots & \Rightarrow \int X_{n+1} \geq \int X_n \geq \int X_{n-1} \cdots \\ X \geq 0 & \Rightarrow \int X \geq 0 \end{array}\right\}$$

If $X_n \uparrow$ seq which is $\geq 0 \Rightarrow \left(\int X_n\right)\uparrow$ & $\left(\int X_n\right) \geq 0$

Let $a_n \equiv \int X_n$

a_n is $\uparrow$ $\geq 0 \Rightarrow \{a_n\}$ has a limit $\in [0, \infty]$

Let $a = \lim_{n\to\infty} \int X_n \, d\mu \leq \int X \, d\mu$ $\left\{\begin{array}{l} X_n \uparrow X \Rightarrow \therefore X_n \leq X \\ \therefore \int X_n \leq \int X \end{array}\right.$

We want to show $\lim_{n\to\infty} \int X_n \, d\mu = \int X \, d\mu$

$\therefore$ we need to show $\lim_{n\to\infty} \int X_n \geq \int X \, d\mu$

We will show $\theta \int X \, d\mu \leq \lim_{n\to\infty} \int X_n \, d\mu \quad \forall \theta \in [0,1]$

and so $\lim_{\theta\to 1} \theta \int X \, d\mu = \int X \, d\mu \leq \lim_{n\to\infty} \int X_n \, d\mu = a$

Now $\int X \, d\mu = \sup\left\{ \int Y \, d\mu \; ; \; 0 \leq Y \leq X \right.$ where Y is a simple +ve function
by definition

$\therefore$ we will show $\theta \int Y \, d\mu \leq a \quad \forall \; 0 \leq \theta \leq 1$ and $0 \leq Y \leq X$ where
Y is simple, positive function

Let $A_n = \{\omega \in \Omega \mid \theta Y(\omega) \leq X_n(\omega)\}$

$\bigcup_{n=1}^{\infty} A_n = \Omega$ and $\{A_n\}\uparrow$ [since $X_n \uparrow$] (done in

$$\theta \int Y I_{A_n} \, d\mu = \int \theta Y I_{A_n} \, d\mu = \int_{A_n} \theta Y \, d\mu \leq \int_{A_n} X_n \, d\mu \leq \int X_n$$

because $X_n \uparrow X$ $\quad \int X_n \leq \lim \int X_n$ $\qquad \leq \lim_{n\to\infty} \int X_n \, d\mu$

$\therefore \quad \theta \int y I_{A_n} \, d\mu \leq \lim_{n\to\infty} \int X_n \, d\mu$

Now $\int y I_{A_n} \, d\mu = \sum_{j=1}^{n} y_j \, \mu(B_j A_n)$ $\quad \left\{ \begin{array}{l} \because y = \sum_{j=1}^{n} y_j I_{B_j} \\ y I_{A_n} = \sum_{j=1}^{n} y_j I_{A_n B_j} \end{array} \right\}$

Also $\lim_{n\to\infty} \mu(B_j A_n) = \mu\left(\lim_{n\to\infty} B_j A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} B_j A_n\right)$ $\quad \left[\begin{array}{l} \{A_n\} \uparrow \text{ seq} \\ \therefore \{B_j A_n\}_n \uparrow \text{ also} \end{array} \right.$

$= \mu\left(B_j \bigcup_{n=1}^{\infty} A_n\right)$

$= \mu(B_j)$ $\quad \left\{ \because \bigcup_{n=1}^{\infty} A_n = \Omega \right\}$

$\therefore \lim_{n\to\infty} \int \theta \, y . I_{A_n} = \theta \sum' y_j \, \mu B_j = \theta \int y \, d\mu$

<u>Step 2</u> $\quad X \geq 0, \; Y \geq 0$

Now let $X_n \uparrow X$ st $\{X_n\}_n$ is simple, ≥ 0

$Y_n \uparrow Y$ st $\{Y_n\}_n$ is simple, ≥ 0

$\therefore (X_n + Y_n) \uparrow (X+Y)$

$\int X + \int Y = \lim_{n\to\infty} \int X_n + \lim_{n\to\infty} \int Y_n$ $\quad$ by MCT

$= \lim_{n\to\infty} \left[\int X_n + \int Y_n \right]$

$= \lim_{n\to\infty} \left(\int X_n + Y_n \right)$ $\quad \left\{ \begin{array}{l} X_n \,\&\, Y_n \text{ are simple} \\ \text{funct}^{\underline{n}} \text{ so use Step 1} \end{array} \right.$

$= \int (X+Y)$ $\quad \left\{ \text{by MCT since } (X_n+Y_n) \uparrow (X+Y) \right.$

Lemma | Fatou's Lemma : If $X \geq 0$ $\int \underline{\lim}\, X_n \, d\mu \leq \underline{\lim} \int X_n d\mu$.

Theorem | The dominated convergence theorem

Suppose a seq $\{X_n\}$ is st $|X_n| \leq Y$ $\forall n$ for some funct $Y \in \mathcal{L}_1 \equiv \{g \mid \int |g| \, d\mu < \infty\}$ $[\Rightarrow \int Y d\mu < \infty]$

If either cond^n holds

(i) $X_n \xrightarrow{a.e} X$

or (ii) $X_n \xrightarrow{\mu} X$

then

$$\boxed{\int |X_n - X| \, d\mu \xrightarrow[n\to\infty]{} 0}$$

NOTE $\int |X_n - X| \, d\mu \longrightarrow 0 \Rightarrow \int X_n - \int X \xrightarrow[n\to\infty]{} 0$

$\left| \int (X_n - X) \right| \longrightarrow 0 \iff \left| \int X_n - \int X \right| \longrightarrow 0$

$\Longrightarrow \int X_n - \int X \longrightarrow 0$.

and $\int |X_n - X| \, d\mu \geq \left| \int (X_n - X) \right|$

So the DCT is stronger than MCT

Proof a) $X_n \xrightarrow{a.e} X \Longrightarrow \int |X_n - X| \, d\mu \longrightarrow 0$

Define $Z_n = |X_n - X| \geq 0$

$Z_n \xrightarrow{a.e} 0 \Rightarrow \underline{\lim}\, Z_n = 0$

We will show $0 \leq \underline{\lim} \int Z_n \, d\mu \leq \overline{\lim} \int Z_n \, d\mu \leq 0$

Now $\underline{\lim}\, Z_n = 0 \Rightarrow \int \underline{\lim}\, Z_n = 0$

$0 = \int \underline{\lim}\, Z_n \leq \underline{\lim} \int Z_n$ [by Fatou's lemma

Now

$Z_n = |X_n - X| \leq |X_n| + |X|$

$\leq Y + Y$ $\left\{ \because \text{since } X_n \xrightarrow{a.e} X \text{ \& } |X_n| \leq Y \right.$

ie $Z_n \leq 2Y$

$\therefore 2Y - Z_n \geq 0$

$\therefore \int \underline{\lim}\,(2Y - Z_n) \leq \underline{\lim} \int (2Y - Z_n)$ Applying Fatou's lemma

$\Rightarrow \int \left(2Y - \underline{\lim}\, Z_n\right) \leq \underline{\lim}\left[\int 2Y + \int(-Z_n)\right]$

$\Rightarrow \int \left(2Y - \underline{\lim}\, Z_n\right) \leq \int 2Y + \underline{\lim} \int (-Z_n)$

$\int 2Y \leq \int 2Y + \underline{\lim}\left(-\int (Z_n)\right)$ $\left\{ \begin{array}{l} \because Z_n = |X_n - X| \xrightarrow{a.e} 0 \\ \underline{\lim}\, Z_n = 0 \end{array} \right.$

$0 \leq \underline{\lim}\left(-\int Z_n\right)$ $\left\{ \begin{array}{l} \int Y \cdot d\mu < \infty \text{ so we} \\ \text{can cancell} \end{array} \right.$

$0 \leq -\overline{\lim}\left(\int Z_n\right)$

$0 \geq \overline{\lim} \int Z_n$

Also $\underline{\lim} \int Z_n \leq \overline{\lim} \int Z_n$ always

$\therefore 0 \leq \underline{\lim} \int Z_n \leq \overline{\lim} \int Z_n \leq 0$

$\therefore \int Z_n \xrightarrow[n \to \infty]{} 0$

ie $\int |X_n - X| \xrightarrow[n \to \infty]{} 0$

Result 1 $\{X_n\}_n$ is a seq st $\boxed{X_n \overset{a.e}{\geq} 0}$ * $\forall n$ then

$$\boxed{\int \sum_{n=1}^{\infty} X_n \, d\mu = \sum_{n=1}^{\infty} \int X_n \, d\mu}$$

ie $$\boxed{\int \lim_{n\to\infty} \sum_{k=1}^{n} X_n \, d\mu} = \lim_{n\to\infty} \sum_{k=1}^{n} \int X_k \, d\mu$$

Proof: let $Z_n = \sum_{k=1}^{n} X_k$

Now since $X_n \geq 0 \ \forall n \Rightarrow Z_n$ is $\uparrow$ & $Z_n \geq 0$

$$\lim_{n\to\infty} Z_n = \lim_{n\to\infty} \sum_{k=1}^{\infty} X_k = \sum_{k=1}^{\infty} X_k \equiv Z$$

ie $Z_n \longrightarrow Z$

By MCT $\int Z_n \xrightarrow[n\to\infty]{} \int Z$

$$\Rightarrow \lim_{n\to\infty} \int Z_n = \int Z$$

$$\lim_{n\to\infty} \left(\int \sum_{k=1}^{n} X_k \right) = \boxed{\int \sum_{k=1}^{\infty} X_k}$$

$$\lim_{n\to\infty} \left(\sum_{k=1}^{n} \int X_k \right) = \int \lim_{n\to\infty} \sum_{k=1}^{n} X_k$$

Q.E.D

THE ABSOLUTE CONTINUITY OF THE INTEGRAL

Let $X \in \mathcal{L}_1$ $\left[\int |X| < \infty\right]$ then for any set A st
$\mu(A) \longrightarrow 0$ then $\int_A |X| \, d\mu \longrightarrow 0$

$$\int |X| \, d\mu = \int |X| \, I_{[|X| \leq n]} + \int |X| \, I_{[|X| > n]} \, d\mu \quad \text{for } n \in \mathbb{N}$$

$[|X| \leq n] \uparrow \Omega$ as $n \longrightarrow \infty$

ie $\{\omega \in \Omega \text{ st } |X(\omega)| \leq n\} \uparrow \Omega$ as $n \to \infty$

$\Rightarrow \quad |X| \, I_{[|X| \leq n]} \uparrow |X| \qquad \{\text{since as } n \to \infty \ I_{[|X| \leq n]} \uparrow I_{\Omega} = 1$

$$\lim_{n \to \infty} \int |X| \, I_{[|X| \leq n]} = \int |X| \qquad \text{by MCT}$$

Also $\lim \int |X| \, d\mu = \int |X| \, d\mu = \lim \left(\int |X| I_{[|X| \leq n]} + \int |X| \, I_{[|X| > n]} \right)$

ie $\int |X| \, d\mu = \lim \int |X| \, I_{[|X| \leq n]} + \lim \int |X| \, I_{[|X| > n]}$

$$\Rightarrow \lim_{n \to \infty} \int |X| \, I_{[|X| > n]} = 0.$$

ie $\forall \ \varepsilon > 0 \ \exists \ n_\varepsilon$ st $\int |X| \, I_{[|X| > n]} \leq \frac{\varepsilon}{2} \quad \forall \ n \geq n_\varepsilon$

Now let A be any arbitrary set st $\mu(A) \longrightarrow 0$
wlg $\mu(A) \leq \frac{\varepsilon}{2 n_\varepsilon}$

$$\int_A |x| \cdot d\mu = \int_A |x| I_{[|x| \le n_\varepsilon]} + \int_A |x| I_{[|x| > n_\varepsilon]}$$

$$\le \int_A |x| I_{[|x| \le n_\varepsilon]} + \int |x| I_{[|x| > n_\varepsilon]}$$

$$\therefore \int_A |x| \cdot d\mu \le \int_{A \cap [|x| \le n_\varepsilon]} |x| \cdot d\mu + \frac{\varepsilon}{2}$$

$$\le n_\varepsilon \mu(A) + \frac{\varepsilon}{2}$$

$\left\{ \begin{array}{l} \text{since } |x| \le n_\varepsilon \\ \int |x| I_{[A \cap |x| \le n_\varepsilon]} \le \end{array} \right.$

$$\le n_\varepsilon \frac{\varepsilon}{2 n_\varepsilon} + \frac{\varepsilon}{2}$$

$$\therefore \int_A |x| \cdot d\mu \le \varepsilon \qquad \forall \varepsilon > 0$$

$$\therefore \int_A |x| \cdot d\mu \longrightarrow 0$$

$$\begin{array}{ccc} (\Omega, \mathcal{A}) & \xrightarrow{X} & (\Omega', \mathcal{A}') \\ & \searrow^{g\circ X} & \downarrow g \\ & & (\overline{\mathbb{R}}, \overline{\mathcal{B}}) \end{array}$$

$\boxed{\text{Theorem of the Unconscious Statistician}}$

$X : (\Omega, \mathcal{A}, \mu) \longrightarrow (\Omega', \mathcal{A}')$ be a meas funct^n

$g : (\Omega', \mathcal{A}') \longrightarrow (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ is another meas funct^n

$g\circ X : (\Omega, \mathcal{A}, \mu) \longrightarrow \overline{\mathbb{R}}, \mathcal{B},$

We can induce a measure $\mu_X(A') \equiv \mu(X^{-1}(A')) \quad \forall A' \in \mathcal{A}'$ for the measurable space $(\Omega', \mathcal{A}')$

$\therefore$ We actually have $g : (\Omega', \mathcal{A}', \mu_X) \longrightarrow (\overline{\mathbb{R}}, \mathcal{B}, \mu_{g(x)})$
where $\mu_{g(x)}$ is the measure induced by 'g'

Then

1) $\mu_{g(x)}$ is fully determined by μ_X

2) $$\int_{X^{-1}(A')} g(X(\omega))\, d\mu(\omega) = \int_{A'} g(x)\, d\mu_X(x) \qquad \forall A' \in \mathcal{A}$$

Take $A' = \Omega'$ then

$$\int_{\Omega} g(X(\omega))\, d\mu(\omega) = \int_{\Omega'} g(x)\, d\mu_X(x)$$

ie $$\int g\circ X \, d\mu = \int g(x)\, d\mu_X(x)$$

Remark: $X : (\Omega, \mathcal{A}, P) \longrightarrow (\mathbb{R}, \mathcal{B}, P_X)$

$g : (\mathbb{R}, \mathcal{B}, P_X) \longrightarrow (\overline{\mathbb{R}}, \overline{\mathcal{B}}, P_{g(x)})$

then
$$\int g\circ X \cdot dP = \int g(x)\, dP_X(x)$$

Now $P_X((-\infty, x]) = P(X^{-1}(-\infty, x])$ {by def of indu

$= P(\{\omega \in \Omega : X(\omega) \leq x\})$

$= P(X \leq x)$

$\equiv F(x)$ some distⁿ functⁿ

If $P_X(x) = P_X((-\infty, x])$ then

$$\int g \circ X \, dP = \int g(x) \, dP_X(x) = \int g(x) \, dF(x)$$

$$= \int g(x) f(x) \, dx \quad [\text{If } F(\cdot) \text{ is diff}$$

So all we need is the form of $g(\cdot)$ and the distⁿ func of X namely $P_X(x) = F(x)$. we do not need distⁿ of $g \circ X$

Proof (1) To show $\mu_{g(X)}$ is fully determined by μ_X

Now since X and g are both measurable $\Rightarrow g \circ X$ is meas.

$$\mu_{g \circ X}(B) = \mu((g \circ X)^{-1}(B)) \quad \forall\, B \in \mathcal{B}$$

$$= \mu(X^{-1} \circ g^{-1}(B))$$

$$= \mu(X^{-1}(g^{-1}(B)))$$

$$= \mu_X(g^{-1}(B))$$ [by def of induced mea

(2) To show $\int g(X(\omega))\, d\mu(\omega) = \int g(x)\, d\mu_X(x)$

(a) Assume $g = I_{A'}$ for some $A' \in \mathcal{A}'$

To show $\int I_{A'}(X) \cdot d\mu = \int I_{A'}\, d\mu_X$

Now $I_{A'}(X)(\omega) = (I_{A'} \circ X)(\omega)$

$$= I_{A'}(X(\omega)) = \begin{cases} 1 & \omega \text{ is st } X(\omega) \in A' \\ 0 & \end{cases}$$

$$(I_{A'}(X))(\omega) = \begin{cases} 1 & \text{if } \omega \in X^{-1}(A') \\ 0 & \text{ow} \end{cases}$$

$$= I_{X^{-1}(A')}(\omega)$$

$\therefore \int I_{A'}(X) \cdot d\mu = \int I_{X^{-1}(A')}\, d\mu$

$= \mu(X^{-1}(A'))$ {by definition of integral

$= \mu_X(A')$ {by def of induced meas.

$= \int I_{A'} \cdot d\mu_X$

$\therefore \int g \circ X\, d\mu = \int g\, d\mu_X$

Rest of the proof in notes

EQUALITY OF THE REIMANN-STIELTJES & LEBESGUE-STIELTJES INTE

<u>Definition</u>

Let $(\mathbb{R}, \hat{\mathcal{B}}_\mu, \mu)$ denote a measurable space with μ a
the L-S measure

$g: \mathbb{R} \longrightarrow \mathbb{R}$ then $\int g \cdot d\mu$ is called the L-S

Note $\int g(x)\, dF(x)$ is called the Reimann Stieltjes integral where F is the cdf corresponding to μ.

$\boxed{\text{Theorem}}$: If 'g' is continuous $\int_a^b g \cdot d\mu = \int_a^b g\, dF$

<u>Proof</u> The R-S integral is defined as follows

Let $a \equiv x_{n_0} < x_{n_1} < \cdots < x_{n_k} < \cdots < x_{n,n} \equiv b$
be a partition of $[a,b]$ st $\text{mesh}_n \equiv \max\limits_{1 \le k \le n} (x_{n_k} - x_{n_{k-1}})$

then

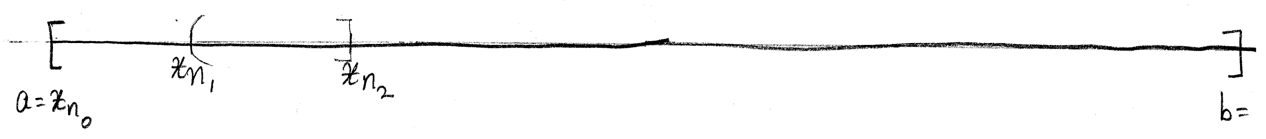
$$\sum_{k=1}^{n} g(x^*_{n_k})\,(x_{n_k} - x_{n_{k-1}}) \longrightarrow \text{R-Integral} \int_a^b g(x)\, dx$$

$x_{n_{k-1}} < x^*_{n_k} \le x_{n_k}$

$$\sum_{k=1}^{n} g(x^*_{n_k})\,(F(x_{n_k}) - F(x_{n_{k-1}})) \xrightarrow[n \to \infty]{} \text{R-S Integral} \int_a^b g\, dF$$

$x_{n_{k-1}} < x^*_{n_k} \le x_{n_k}$

a) Let $g_n(x) = \sum_{k=1}^{n} g(x^*_{n_k})\, I_{(x_{n_{k-1}}, x_{n_k}]}(x)$



$g_n(x) = g(x^*_{n_{k_0}})$ for some k_0 where the indicator is $\neq 0$

$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} g(x^*_{n_{k_0}})$$

$$= g\left(\lim_{n\to\infty}(x^*_{n_{k_0}})\right) \quad \{\text{since } g \text{ is continuous}$$

$$= g(x) \quad \text{uniformly}$$

ie $g_n(x) \longrightarrow g(x)$ uniformly

? Now since 'g' is continuous on the compact set $[a,b]$ then $\exists$ an M st $\forall x$, $|g_n(x)| \leq M$

$$\therefore \int_a^b M \cdot d\mu = M\,\mu([a,b]) \quad \text{(by DCT)}$$

Now L-S integ $= \int_a^b g\,d\mu$

$$= \lim_{n\to\infty} \int_a^b g_n\,d\mu \quad \text{(by DCT)}$$

$$= \lim_{n\to\infty} \int_a^b \sum_{k=1}^{n} g(x^*_{n_k})\, I_{(x_{n_{k-1}}, x_{n_k}]}\,d\mu$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \int_a^b g(x^*_{n_k})\, I_{(x_{n_{k-1}}, x_{n_k}]}\,d\mu$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} g(x^*_{n_k}) \int_a^b I_{(x_{n_{k-1}}, x_{n_k}]}\,d\mu$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} g(x^*_{n_k})\, \mu\left((x_{n_{k-1}}, x_{n_k}]\right)$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} g(x^*_{n_k}) \left(F(x_{n_k}) - F(x_{n_{k-1}})\right)$$ { by correspon… theorem f…

$$= \int g(x)\, dF(x)$$ (by def of R-S integral.

INEQUALITIES

L_p Spaces : $\mathcal{L}_p = \{X : \int |X|^p < \infty\}$ for some $p>0$

PROPOSITION 1 : Relationship b/w different L_p spaces

Let $\mu(\Omega) < \infty$ then $\mathcal{L}_s \subset \mathcal{L}_r$ $0<r<s$

Proof : Let $X \in \mathcal{L}_s \Rightarrow \int |X|^s < \infty$

We want to show $\int |X|^r < \infty$

We always have ; for any $a>0$ and $0<r<s$; $a^r \le 1 + a^s$

Take $a = |X|$ and then integrate

$\therefore |X|^r \le 1 + |X|^s$

$\therefore \int |X|^r \le \int 1 + \int |X|^s$

$< \infty$ [$\because \int |X|^s < \infty$ & $\int 1 = \mu(\Omega)$

Example : $\mathcal{L}_2 \subset \mathcal{L}_1$

ie $\{X : \int X^2 \, d\mu < \infty\} \subset \{X : \int X < \infty\}$

By cauchy schwartz $E|X \cdot 1| \le [E(|X|^2)]^{1/2} [E(|Y|^2)]^{1/2}$

ie $\int |X| \cdot d\mu \leq \left(\int |X|^2\right)^{1/2} \left(\int 1\right)^{1/2}$

<u>NOTE</u> Result proved in $\mathcal{L}_1$ is stronger than one proved in $\mathcal{L}_2$

Now suppose we want to measure the distance b/w 2 functions f and g (say). The most used distance is the $\mathcal{L}_p$ norm $\|\cdot\|_p$.

= The $\mathcal{L}_r$ norm of X is $\|X\|_r \equiv \left(\int |X|^r\right)^{1/r} = \left[E(|X|^r)\right]^{1/r}$

$\therefore$ to show 2 functions are close if $\|f-g\|_r < \varepsilon$

EF $\|\cdot\|$ is a norm if

1) $\|a\| = 0$ iff $a = 0$
2) $\|ca\| = |c| \, \|a\|$ for any $c \in \mathbb{R}$
3) $\|a+b\| \leq \|a\| + \|b\|$

CK 1) & 2) hold for the $\mathcal{L}_r$-norm
3) holds by Minkowski's inequality proof

ITY 1 $E(|X+Y|^r) \leq 2^{r-1} E(|X|^r) + 2^{r-1} E(|Y|^r) \quad \forall\, r \geq 1$

eg: $E[(X+Y)^2] \leq 2E(X^2) + 2E(Y^2)$

$\because \quad (X+Y)^2 \leq 2X^2 + Y^2$

R's $E(|XY|) \leq \left[E(|X|^r)\right]^{1/r} \left[E(|Y|^s)\right]^{1/s}$ for any $r > 1$ and any $s > 0$ st $\frac{1}{r} + \frac{1}{s} = 1$

ie $\|XY\|_1 \leq \|X\|_r \, \|Y\|_s$

Proof: Young's inequality $|ab| \leq \frac{|a|^r}{r} + \frac{|b|^s}{s}$ $\forall$ a, b and $r>1$, s. st $\frac{1}{r}+\frac{1}{s}=1$

Let $f(x) = e^x$ is a convex function $\{$since $f''(x) > 0\}$

$\Rightarrow$ $f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y)$; $0 \leq \alpha \leq 1$

Let $\alpha = \frac{1}{r}$ & $1-\alpha = \frac{1}{s}$

Let $x = r\log|a|$ and $y = s\log|b|$

then $\exp\left\{\frac{r\log|a|}{r} + \frac{s\log|b|}{s}\right\} \leq \frac{1}{r}\exp\{r\log|a|\} + \frac{1}{s}\exp\{s\log|$

$\Rightarrow$ $\exp\{\log(|a||b|)\} \leq \frac{|a|^r}{r} + \frac{|b|^s}{s}$

$\Rightarrow$ $|a|\cdot|b| \leq \frac{|a|^r}{r} + \frac{|b|^s}{s}$ ——(*)

Now let $a = \frac{|X|}{\|X\|_r}$ and $b = \frac{|Y|}{\|Y\|_s}$

Now $E\left(\frac{|X|}{\|X\|_r} \frac{|Y|}{\|Y\|_s}\right) = \frac{1}{\|X\|_r\|Y\|_s} E(|X|\cdot|Y|)$

$$\leq \frac{1}{r} E\left[\frac{|X|^r}{(\|X\|_r)^r}\right] + \frac{1}{s}E\left[\frac{|Y|^s}{\|Y\|_s^s}\right]$$

$$= \frac{1}{r}\frac{E(|X|^r)}{E(|X|^r)} + \frac{1}{s}\frac{E(|Y|^s)}{E(|Y|^s)}$$

$$= \frac{1}{r} + \frac{1}{s}$$

$\therefore \quad E\left(|X| \cdot |Y|\right) \leq \|X\|_r \cdot \|X\|_s$

Cauchy-Schwartz Ineq: Use $r = s = 2$ in Hölders inequality

$\|X\|_r \uparrow$ in r $\forall\ r > 0$ $\quad [\text{and } \mu(\Omega) < \infty]$

ie $\quad \|X\|_1 \leq \|X\|_2 \leq \|X\|_3 \leq \cdots$

[Use to show $\|X\|_r < \infty$ is enough to show $\|X\|_s < \infty$; $r < s$]

VEQ Let $g \geq 0$, $\uparrow$ on $[0, \infty)$ and even function then for all measurable functions X we have

$$\mu\left(|X| \geq \lambda\right) \leq \frac{E[g(X)]}{g(\lambda)} \qquad \forall\ \lambda > 0$$

[Used to show convergene in measures and convergence almost surely]

ie $$\mu\left(|X| \geq \lambda\right) \leq \frac{\|g(X)\|_1}{g(\lambda)}$$

Pf $$E[g(X)] = \int g(X)\, d\mu = \int_{[|X| \geq \lambda]} g \circ X \, d\mu + \int_{[|X| < \lambda]} g \circ X \, d\mu$$

$$\geq \int_{[|X| \geq \lambda]} g \circ X \, d\mu \qquad \left\{ \because g \geq 0 \Rightarrow \int g \geq 0 \right.$$

$$\geq \int_{[|X| \geq \lambda]} g(\lambda)\, d\mu$$

$$= g(\lambda) \int_{[|X| \geq \lambda]} d\mu$$

$$= \quad g(\lambda)\, \mu([|X| > \lambda])$$

ie $\quad E[g(X)] \geq g(\lambda) \quad \mu([|X| > \lambda])$

Example 1 $\quad$ Markov's Inequality $\qquad \mu([|X| \geq \lambda]) \leq \dfrac{E(|X|}{\lambda^r}$

Use $g(x) = |x|^r$

Example 2 $\quad$ Chebyshev's ineq $\qquad P(|X - \mu| \geq \lambda) \leq \dfrac{Var(X)}{\lambda^2}$

Use $g(x) = x^2$ and $X \equiv X - \mu$ in Basic inequ

1) $\xrightarrow{a.c} \Rightarrow \xrightarrow{\mu} \Rightarrow \xrightarrow{d}$

2) $X_n \longrightarrow X \Rightarrow g(X_n) \longrightarrow g(X)$ for any g continuous

any type of convg

3) $X_n \xrightarrow{L_r} X$ ie $E|X_n - X|^r \longrightarrow 0 \Rightarrow E(X_n) \longrightarrow E(X)$