$$X^{-1}(\mathcal{C}) \subset \underbrace{X^{-1}(\sigma(\mathcal{C}))}_{\sigma\text{-field}} \Rightarrow \sigma(X^{-1}(\mathcal{C})) \subset X^{-1}[\sigma(\mathcal{C})]$$

#1 X is $\mathcal{B}$-$\mathcal{A}_0$ measurable when $X^{-1}(B) \in \mathcal{A}_0 \quad \forall\ B \in \mathcal{B}$

X is $\mathcal{A}_1$ measurable $\longleftrightarrow$ X is $\mathcal{B}$-$\mathcal{A}_1$ meas

$$X: ([0,1], \mathcal{B}_{[0,1]}) \longrightarrow (\mathbb{R}, \mathcal{B})$$

(a) $X_1(\omega) = 2\omega$ if $X_1^{-1}(B) \notin \mathcal{A}_0$ for some $B \in \mathcal{B}$ then X_1 is not

Now $X_1^{-1}[0,1] = [0, 1/2] \notin \mathcal{A}_0$

$\therefore X_1$ is not $\mathcal{B}$-$\mathcal{A}_0$ measurable

(b) Is $X_2^{-1}(\mathcal{B})$ a σ-field

$$X_2^{-1}(\mathcal{B}) = X_2^{-1}[\sigma(\mathcal{C})] = \sigma[X_2^{-1}(\mathcal{C})] \quad \{\text{true for any } X$$

where $\mathcal{C} = \{(-\infty, a] \ ; a \in \mathbb{R}\}$

$$X_2(\omega) = \frac{\pi}{2} I_{[0,\frac{1}{5}]}(\omega) + 3 I_{[\frac{3}{4},1]}(\omega)$$

$$X_2(\omega) = \begin{cases} 0 & \omega \in (1/5, 3/4) \\ \pi/2 & \omega \in [0, 1/5] \\ 3 & \omega \in [3/4, 1] \end{cases}$$

$$X_2^{-1}(\mathcal{C}) = X_2^{-1}((-\infty, a]) = \begin{cases} \phi & \text{if } a < 0 \\ (1/5, 3/4) & \frac{\pi}{2} > a \geq 0 \\ [0, 3/4) & 3 > a \geq \pi/2 \\ \Omega & a \geq 3 \end{cases}$$

$$X_2^{-1}(\mathcal{B}) = \sigma\{\phi, (1/5, 3/4), [0, 3/4), \Omega\}$$

(c) $\mathcal{A}_1 = \sigma(\{\phi, [0, 1/3], (1/3, 2/3], (2/3, 1]\})$.

generators of $\mathcal{A}_1$ form a partition of Ω (measurable partition)

All functions X which are $\mathcal{A}_1$ measurable must be of the form

$$\mathcal{D} = \{X \mid \alpha_1 I_{[0,1/3]} + \alpha_2 I_{(1/3, 2/3]} + \alpha_3 I_{(2/3,1]} = X\}$$

Proof by contradiction

If $X \in \mathcal{D}$ and $X(u) = x_1$ & $X(v) = x_2$ where $x_1 < x_2$

and $u, v \in [0, 1/3]$

Now $X^{-1}((-\infty, x_1])$ & $X^{-1}((-\infty, x_2]) \in \mathcal{A}_1$ {X is meas.}

$[0, 1/3] \in \mathcal{A}_1$ { $\mathcal{A}_1$ is σ-field }

$\therefore$ $\left.\begin{array}{l} [0,1/3] \cap X^{-1}((-\infty, x_1]) \\ [0,1/3] \cap X^{-1}((-\infty, x_2]) \end{array}\right\}$ are disjoint & $\in \mathcal{A}_1$

But we now have 2 disjoint sets in $\mathcal{A}_1$ which cannot be since you cannot express them as unions or intersections of elements of $\mathcal{A}_1$

$\therefore$ we have a contradiction and $X \notin \mathcal{D}$

ie X must be const over each of the intervals which generate $\mathcal{A}_1$