INDUCED MEASURE

$X:(\Omega,\mathcal{A},\mu) \longrightarrow (\Omega',\mathcal{A}')$ and X is $\mathcal{A}'-\mathcal{A}$ meas $\Rightarrow \mu_X(A') \equiv \mu(X^{-1}(A'))$, $A'\in\mathcal{A}'$

is an induced meas on $(\Omega',\mathcal{A}')$

(i) $\mu_X(\Omega') = \mu(X^{-1}(\Omega)) = \mu(\Omega)$

(ii) $\mu_X(\phi) = \mu(X^{-1}(\phi)) = \mu(\phi) = 0$

(iii) $\mu_X\left(\sum_{n=1}^{\infty} A_n'\right) = \mu\left(X^{-1}\left(\sum_{n=1}^{\infty} A_n'\right)\right) = \mu\left(\sum_{n=1}^{\infty} X^{-1}(A_n')\right) = \sum_{n=1}^{\infty}\mu(X^{-1}(A_n'))$

$$= \sum_{n=1}^{\infty}\mu_X(A_n')$$

<u>NOTE</u>: If μ is a probability meas so is μ_X

X can be viewed as a $\mathcal{A}'-\mathcal{F}(X)$ meas funct^n from $(\Omega,\mathcal{F}(X),\mu)\longrightarrow(\Omega',\mathcal{A}',\mu_X)$

<u>PROP</u> (The form of an $\mathcal{F}(X)$ meas funct^n)

$X:(\Omega,\mathcal{A},\mu)\longrightarrow(\mathbb{R},\mathcal{B})$ is $\mathcal{B}-\mathcal{A}$ meas

$Y:(\Omega,\mathcal{F}(X))\longrightarrow(\mathbb{R},\mathcal{B})$ is $\mathcal{B}-\mathcal{F}(X)$ meas $\quad [\mathcal{F}(X)\subset\mathcal{A}]$

Then $\exists$ $g:(\mathbb{R},\mathcal{B})\longrightarrow(\mathbb{R},\mathcal{B})$ which is measurable and $Y = g\circ X$

<u>Proof</u> <u>Step 1</u> $Y = I_D$ for some $D\in\mathcal{F}(X) \equiv \{X^{-1}(B) : B\in\mathcal{B}\}$

$$Y(\omega) = I_D(\omega) = I_{X^{-1}(B)}(\omega) = \begin{cases}1 & \text{if } \omega\in X^{-1}(B)\\ 0 & \text{ow}\end{cases}$$

$$= \begin{cases}1 & \text{if } \omega\in\{\omega\in\Omega \mid X(\omega)\in B\}\\ 0 & \text{ow}\end{cases}$$

$$= \begin{cases}1 & \text{if } X(\omega)\in B\\ 0 & \text{ow}\end{cases}$$

$$= I_B(X(\omega))$$

ie $Y = (I_B\circ X)$

Also I_B for some $B\in\mathcal{B}$ is measurable $[(\mathbb{R},\mathcal{B})\to(\mathbb{R},\mathcal{B})]$

Here $g\equiv I_B$

Step 2 : Y is a simple funct^n ie $Y = \sum_{i=1}^{n} x_i I_{D_i}$; $D_i \in \mathcal{F}(X)$, $x_i \in \mathbb{R}$

$$Y(\omega) = \sum_{i=1}^{n} x_i I_{D_i}(\omega) = \sum_{i=1}^{n} x_i I_{X^{-1}(B_i)}(\omega) = \sum_{i=1}^{n} x_i I_{B_i}(X(\omega)) \equiv g \circ X$$

Since all $B_i \in \mathcal{B}$ $g = \sum_{i=1}^{n} x_i I_{B_i}$ is a $\mathcal{B}$-$\mathcal{B}$ meas funct^n (all simple funct^{n}s are)

Step 3 Let $Y \geq 0$ and Y is $\mathcal{F}(X)$ measurable

$\therefore \exists$ a $\uparrow$ seq of funct^n $Y_n \geq 0$ which are also $\mathcal{F}(X)$ meas st

$Y_n \uparrow Y$ ie $\lim_{n\to\infty} Y_n = Y$

For each Y_n from step 2 $\Rightarrow$ $Y_n = g_n \circ X$ with $g_n \uparrow$ simple $\mathcal{B}$ meas

$\therefore$ $Y = \lim Y_n = \lim_{n\to\infty} g_n(X)$

$= g(X)$ $\left\{ \begin{array}{l} g_n \text{ is a seq of } \mathcal{B} \text{ meas funct} \\ \Rightarrow g = \lim g_n \text{ is also meas} \end{array} \right.$

Step 4 If Y is any arbitrary funct^n $Y = Y^+ - Y^-$ where $Y^+, Y^- \geq 0$

Also if Y is $\mathcal{F}(X)$ measurable $\rightarrow$ Y^+, Y^- are also $\mathcal{F}(X)$ meas.

$\therefore$ $Y^+ = g^+ \circ X$ and $Y^- = g^- \circ X$ for some g^+ and g^-

which are $\mathcal{B}$- measurable

$$Y = Y^+ - Y^- = g^+ \circ X - g^- \circ X = (g^+ - g^-) \circ X$$

$\equiv g \circ X$ $\left\{ g \right.$ is measurable since g^+ & g^- are both measurable.

NOTE Every induced measure on $(\mathbb{R}, \mathcal{B})$ can be uniquely characterized by a df. $F(\cdot)$
$X : (\Omega, \mathcal{A}, \mu) \longrightarrow (\mathbb{R}, \mathcal{B})$ the induced meas $\mu_X(\cdot) = \mu(X^{-1}(\cdot))$ is a L-S meas.
So by the correspondence theorem $\mu_X((a,b]) = F(b) - F(a)$

CONVERGENCE ALMOST EVERYWHERE

- X_n is a seq of meas funct^n on $(\Omega, \mathcal{A}, \mu)$ and X is a meas on $(\Omega, \mathcal{A}, \mu)$

$X_n \xrightarrow{ae} X$ if $\lim_{n\to\infty} X_n(\omega) = X(\omega)$ $\forall\ \omega \in C$ and $\mu(C^c) = 0$

- We say X_n mutually converges a.e to X if $X_n - X_m \xrightarrow[m\wedge n \to \infty]{a.e} 0$

- Convergence set $[X_n \to X] = \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{m\geq n}^{\infty} \left[|X_m - X| < 1/k \right] \in \mathcal{A}$

- Mutual convergence set $[X_m - X_n \to 0] = \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{m\geq n}^{\infty} \left[|X_m - X_n| < 1/k \right] \in \mathcal{A}$

- Divergence set $[X_n \to X]^c = \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} \underbrace{\bigcup_{m\geq n}^{\infty} \left[|X_m - X| \geq 1/k \right]}_{B_{nk}\ \downarrow} \equiv \bigcup_{k=1}^{\infty} A_k$

where $A_k = \bigcap_{n=1}^{\infty} (B_{n_k})$ and $A_k \uparrow$ in k

<u>Proposition</u> Let X_n and X be finite measurable functions

(a) $X_n \xrightarrow{a.e} X \iff \mu\left(\bigcap_{n=1}^{\infty} \bigcup_{m\geq n}^{\infty} \left[|X_m - X_n| \geq \varepsilon \right] \right) = 0 \quad \forall\ \varepsilon > 0$

(b) If $\mu(\Omega) < \infty$ then $X_n \xrightarrow{a.e} X \iff \mu\left(\bigcup_{m\geq n}^{\infty} \left[|X_m - X_n| \geq \varepsilon \right] \right) \to 0 \quad \forall\ \varepsilon > 0$

$\iff \mu\left(\left[\max_{n\leq m\leq N} |X_m - X_n| \geq \varepsilon \right] \right) \leq \varepsilon \quad \forall\ N \geq n \geq n_\varepsilon,\ \forall\ \varepsilon > 0$

CONVERGENCE IN MEASURE

- X_n is a seq of a.e finite meas functions. X is a measurable funct^n with values in $\overline{\mathbb{R}}$

$$X_n \xrightarrow[n\to\infty]{\mu} X \quad \text{if} \quad \mu([|X_n - X| \geq \varepsilon]) \xrightarrow[n\to\infty]{} 0 \qquad \forall\ \varepsilon > 0$$

NOTE: This means X must be finite a.e as $[|X| = \infty] \subset \{[\bigcup_{k=1}^{\infty} |X_k| = \infty]\} \cup \{[|X_n - X| \geq \varepsilon]\}$

- X_n is said to converge mutually in meas.

$$X_n - X_m \xrightarrow{\mu} 0 \quad \text{iff} \quad \mu([|X_m - X_n| \geq \varepsilon]) \xrightarrow[m\wedge n\to\infty]{} 0 \qquad \forall\ \varepsilon > 0.$$

PROP 3.4 If $X_n \xrightarrow{\mu} X$ and $X_n \xrightarrow{\mu} \tilde{X}$ then $X = \tilde{X}$ a.e.

Proof To show $\mu([X \neq \tilde{X}]) = 0$

Consider $[|X - \tilde{X}| \geq 2\varepsilon] \subseteq [|X - X_n| \geq \varepsilon] \cup [|X_n - \tilde{X}| \geq \varepsilon]$

$$\therefore \quad \mu[|X - \tilde{X}| \geq 2\varepsilon] \leq \mu[|X - X_n| \geq \varepsilon] + \mu[|X_n - \tilde{X}| \geq \varepsilon] \qquad \forall \varepsilon > 0$$

$$0 \leq \mu[|X - \tilde{X}| \geq 2\varepsilon] \to 0. \qquad \{\because X_n \xrightarrow{\mu} X \ \&\ X_n \xrightarrow{\mu} \tilde{X}$$

$$\therefore \quad \mu([X \neq \tilde{X}]) = \mu\left(\bigcup_{k=1}^{\infty} [|X - \tilde{X}| \geq 1/k]\right) \leq \sum_{k=1}^{\infty} \mu([|X - \tilde{X}| \geq 1/k]) = 0$$

THEOREM [Relationship b/w $\xrightarrow{\mu}$ & $\xrightarrow{a.e}$] If $X_1, X_2, \ldots$ are measurable and finite a.e

1) $X_n \xrightarrow{a.e} X \iff X_n - X_m \xrightarrow{a.e} 0$

2) $X_n \xrightarrow{\mu} X \iff X_n - X_m \xrightarrow{\mu} 0$

3) ** If $\mu(\Omega) < \infty$ then $X_n \xrightarrow{a.e} X \Rightarrow X_n \xrightarrow{\mu} X$

4) If $X_n \xrightarrow{\mu} X$ then for some subsequence n_k we have $X_{n_k} \xrightarrow{a.e} X$

5) If $\mu(\Omega) < \infty$ then $X_n \xrightarrow{a.e} X \iff$ each subseq n' has a further n'' st $X_{n''} \xrightarrow{a.e} X$

Proof : To show $X_n \xrightarrow{a.e} X \Rightarrow X_n \xrightarrow{\mu} X$ when $\mu(\Omega) < \infty$

Consider $\mu([|X_n - X| \geq \varepsilon]) \leq \mu\left(\bigcup_{m\geq n}^{\infty} [|X_m - X| \geq \varepsilon]\right)$ $\left\{ \because \{|X_n - X| \geq \varepsilon\} \subset \bigcup_{m \geq n}^{\infty} \{|X_m - X| \geq \varepsilon\} \right\}$

Now if $\mu(\Omega) < \infty$ then $X_n \xrightarrow{a.e} X \iff \mu\left(\bigcup_{m\geq n}^{\infty}[|X_m - X| \geq \varepsilon]\right) \longrightarrow 0 \quad \forall \varepsilon > 0$

$$0 \leq \lim_{n \to \infty} \mu([|X_n - X| \geq \varepsilon]) \leq 0$$

$$\therefore \quad \mu([|X_n - X| \geq \varepsilon]) \xrightarrow[n\to\infty]{} 0 \qquad \text{ie} \qquad X_n \xrightarrow{\mu} X$$

NOTE Convergence in meas $\not\Rightarrow$ convergence a.e

Proof We will construct a seq $\{X_n\}$ st $X_n \xrightarrow{\mu} X$ but $X_n \overset{a.e}{\not\to} X$

Let $X_n : (\Omega, \mathcal{A}, \mu) \longrightarrow (\mathbb{R}, \mathcal{B})$ where $\Omega = [0,1)$, $\mathcal{A} = \mathcal{B}_{[0,1)}$, $\mu = \lambda$ (lebesgue meas)

We define the X_n's as follows

$$\left.\begin{aligned}
X_1 &= I_{[0,1)} \\
X_2 &= I_{[0,1/2)} \\
X_3 &= I_{[1/2,1)} \\
X_4 &= I_{[0,1/3)} \\
X_5 &= I_{[1/3,2/3)} \\
X_6 &= I_{[2/3,1)} \\
&\vdots
\end{aligned}\right\} \quad \text{ie} \quad X_{\frac{m(m-1)}{2}+k} = I_{\left[\frac{k-1}{m}, \frac{k}{m}\right)} \; ; \quad \begin{matrix} m \geq 1 \\ k = 1, 2 \cdots m \end{matrix}$$

Define $X = 0$

Step 1 To show $\mu([|X_n - X| > \varepsilon]) \longrightarrow 0 \quad \forall \varepsilon > 0.$ as $n \to \infty$

ie to show $\forall \varepsilon > 0$ and $\delta > 0$ $\exists n_\varepsilon$ st $\mu(\{\omega \in \Omega : |X_n(\omega) - X(\omega)| \geq \varepsilon\}) < \delta$, $\forall n \geq n_\varepsilon$

Now $\mu([|X_n(\omega) - X(\omega)| \geq \varepsilon]) = \mu([|X_n(\omega)| \geq \varepsilon])$ $\{ \because X(\omega) = 0 \ \forall \omega \in \Omega$ by definition

$= \mu([X_n(\omega) = 1])$ $\left\{ \begin{matrix} \because X_n\text{'s are all indicators} \\ X_n \geq \varepsilon \Rightarrow X_n = 1 \end{matrix} \right.$

$\therefore \mu([|X_n(\omega)| > \varepsilon]) = \mu(A_n)$ $\left\{ \text{where } A_n = A_{\frac{m(m-1)}{2}+k} = \left[\frac{k-1}{m}, \frac{k}{m}\right) \text{ and } X_n(\omega) = \begin{cases} 1 & \forall\ \omega \in A_n \\ 0 & \text{ow} \end{cases} \right.$

$\therefore \lim_{n\to\infty} \mu([|X_n - X| \geq \varepsilon]) = \lim_{n\to\infty} \mu(A_n) = 0$ $\left\{ \because \text{lengths of } A_n \downarrow 0 \text{ as } n \to \infty \right.$

$\therefore\ X_n \xrightarrow{\mu} X$ as $n \to \infty$

__Step 2__ We will now show $X_n \overset{a.e}{\not\to} X$ by showing $\lim_{n\to\infty} X_n$ does not exist $\forall\ \omega \in \Omega$

We will show $\overline{\lim}\ X_n \neq \underline{\lim}\ X_n$ $\forall\ \omega \in \Omega$ [Not just on a set N]

Now $\overline{\lim_{n\to\infty}}\ X_n(\omega) = \lim_{n\to\infty} \sup_{m \geq n} X_m(\omega)$

$= \lim_{n\to\infty} (1)$ $\left\{ \because \text{given any } n_\varepsilon\ \exists \text{ some } A_{n_0} \text{ st } \omega \in A_{n_0}\ \&\ I_{A_{n_0}} = 1 \text{ where } n_0 \geq n_\varepsilon \right.$

$= 1$

Also $\underline{\lim_{n\to\infty}}\ X_n(\omega) = \lim_{n\to\infty} \inf_{m \geq n} X_m(\omega)$

$= \lim_{n\to\infty} 0$ $\left\{ \because \text{given any } n_\varepsilon\ \exists \text{ some } A_{n_0} \text{ st } \omega \in A_{n_0} \text{ and } \omega \notin A_k\ \forall\ k \neq n_0 \text{ and } n_0 \geq n_\varepsilon \right.$

$= 0$

$\therefore\ \underline{\lim}\ X_n \neq \overline{\lim}\ X_n \Rightarrow$ limit does not exist $\forall\ \omega \in \Omega$

$\therefore\ X_n \overset{a.e}{\not\to} X$

__Example__ To show if $\mu(\Omega) = \infty$ then $X_n \xrightarrow{a.e} X$ need not $\Rightarrow$ $X_n \xrightarrow{\mu} X$

__Proof__ We will construct a seq $X_n : (\Omega, \mathcal{A}, \mu) \to (\mathbb{R}, \mathcal{B})$ st $X_n \xrightarrow{a.e} X$ but $X_n \overset{\mu}{\not\to} X$

Let $\Omega = [0, \infty)$, $\mathcal{A} = \mathcal{B}_{[0,\infty)}$, $\mu = \lambda$ (lebesgue measure)

Notice that $\mu(\Omega) = \lambda([0,\infty)) = \infty$

Define $X_n = I_{[n, n+1)}$ $n = 0, 1, 2 \ldots$

Define $X = 0$ $\forall\ \omega \in \Omega$

Step 1 to show $X_n(\omega_0) \to X(\omega_0)$ for any arbitrary $\omega_0 \in \Omega$

Consider $\omega_0 \in [0, \infty) \to \exists$ some n_0 st $\omega \in [n_0, n_0+1)$

$$\Rightarrow \quad X_{n_0}(\omega_0) = 1 \quad \text{and} \quad X_n(\omega_0) = 0 \quad \forall\ n \neq n_0$$

$$\lim_{n\to\infty} X_n(\omega_0) = 0 = X(\omega_0) \qquad \forall\ n \geq n_0$$

$$\therefore \quad X_n \xrightarrow{a.e} X$$

Step 2 to show $\mu([|X_n - X| \geq \varepsilon]) \not\to 0$ for some $\varepsilon > 0$

Consider $\mu([|X_n - X| \geq \varepsilon]) = \mu([|X_n| \geq \varepsilon])$

$$= \mu([X_n = 1])$$

$$= \mu([n, n+1))$$

$$= 1 \not\to 0$$

$\therefore$ for any $\varepsilon > 0$, $\mu([|X_n - X| \geq \varepsilon]) \not\to 0$

$$\therefore \quad X_n \not\xrightarrow{\mu} X$$

PROBABILITY, RANDOM VARIABLES

- Let $X : (\Omega, \mathcal{A}, P) \to (\mathbb{R}, \mathcal{B})$ where P is a prob. meas ie $P(\Omega) = 1$ & X is meas then X is called a random variable.

- It is $\mathcal{B}$-$\mathcal{A}$ measurable $\Rightarrow$ X induces a meas $P_X(\cdot) \equiv P(X^{-1}(\cdot))$ on $(\mathbb{R}, \mathcal{B})$ which is called the induced dist^n of X

- Also since P is a prob meas $\Rightarrow$ P_X is also a prob meas and hence a L-S meas. So the correspondence theorem gives us

$$P_X((-\infty, x]) = P([X \leq x]) = F_X(x)$$ for a dist^n funct^n $F_X(\cdot)$ which is $\uparrow$ rt continuous and has $F_X(-\infty) = 0$ and $F_X(\infty) = 1$

- $X \cong F_X$ means the induced dist^n $P_X(\cdot)$ of the r.v X has a dist^n funct^n $F_X(\cdot)$

CONVERGENCE IN DISTRIBUTION

- Let $\{X_n\}$ be a seq of random variables, $X_n \cong F_{X_n}$ and X_0 is a r.v, $X_0 \cong F_{X_0}$

$$X_n \xrightarrow{d} X_0 \quad [\text{or } \mathcal{L}(X_n) \longrightarrow \mathcal{L}(X_0)] \quad \text{if} \quad F_{X_n}(x) \longrightarrow F_{X_0}(x) \quad \forall\ x \in C(F)$$

where $C(F)$ = set of continuity pts of the funct^n F

Theorem : $X_n \xrightarrow{\mu} X \Rightarrow X_n \xrightarrow{d} X$ for $X_n \cong F_{X_n}$ and $X \cong F_X$

Proof : We need to show $F_{X_n}(t) \longrightarrow F_X(t)$ $\forall\ t \in C(F)$

Consider
$$F_{X_n}(t) = \mu([X_n \leq t]) = \mu([X_n + X - X \leq t])$$
$$= \mu([X_n - X + X \leq t + \varepsilon - \varepsilon])$$

Now
$$\left.\begin{array}{l} X_n - X \geq -\varepsilon \\ \text{and} \quad X \geq t+\varepsilon \end{array}\right\} \Rightarrow X_n \geq t$$

$$\therefore \quad [X_n \leq t] \subseteq [X_n - X \leq -\varepsilon] \cup [X \leq t+\varepsilon]$$

$$\therefore \quad F_{X_n}(t) \leq \mu([X_n - X \leq -\varepsilon]) + \mu([X \leq t+\varepsilon])$$

Now $[|X_n - X| \geq \varepsilon] = [X_n - X \geq \varepsilon] \cup [X_n - X \leq -\varepsilon]$

$$\therefore \quad \mu[|X_n - X| \geq \varepsilon] \geq \mu([X_n - X \leq -\varepsilon])$$

$$\therefore \quad F_{X_n}(t) \leq \mu([|X_n - X| \geq \varepsilon]) + \mu([X \leq t+\varepsilon])$$

$$F_{X_n}(t) \leq \mu([|X_n - X| \geq \varepsilon]) + F_X(t+\varepsilon)$$

$$F_{X_n}(t) \leq \varepsilon + F_X(t+\varepsilon) \quad \forall\ n \geq n_1 \quad \{\because X_n \xrightarrow{\mu} X$$

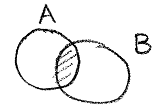
$$\therefore \quad \overline{\lim}\, F_{X_n}(t) \leq F_X(t+\varepsilon) + \varepsilon \qquad \text{--- } (1)$$

Now. $F_{X_n}(t) = \mu_{X_n}((-\infty, t]) = \mu([X_n \le t])$

$$\left.\begin{array}{l} |X_n - X| \le \varepsilon \\ \& \quad X \le t-\varepsilon \end{array}\right\} \Rightarrow X_n \le t \qquad \left\{\begin{array}{l} \because \ |X_n - X| \le \varepsilon \ \& \ X \le t-\varepsilon \\ \Rightarrow \ X_n - X \le \varepsilon \ \& \ X \le t-\varepsilon \end{array}\right\}$$

$A \subseteq B$

$$\therefore \ \mu([X_n \le t]) \ge \mu([|X_n - X| \le \varepsilon \ \cap \ [X \le t-\varepsilon])$$



$$\left\{\begin{array}{l} P(AB) = P(A) - P(AB^c) \\ \ge P(A) - P(B^c) \\ \text{as } P(AB^c) \le P(B^c) \end{array}\right.$$

$$\therefore \quad F_{X_n}(t) \ge \mu([X \le t-\varepsilon]) - \mu([|X_n - X| > \varepsilon])$$

$$F_{X_n}(t) \ge F_X(t-\varepsilon) - \varepsilon \qquad \forall\ n \ge n_2$$

$$\therefore \quad \underline{\lim}\ F_{X_n}(t) \ge F_X(t-\varepsilon) - \varepsilon \qquad \text{— (2)}$$

$$\therefore \quad F_X(t-\varepsilon) - \varepsilon \le \underline{\lim}\ F_{X_n}(t) \le \overline{\lim}\ F_{X_n}(t) \le F_X(t+\varepsilon) + \varepsilon$$

$$\Rightarrow \quad F_X(t) \le \underline{\lim}\ F_{X_n}(t) \le \overline{\lim}\ F_{X_n}(t) \le F_X(t) \qquad \text{for } t \in C(F) \text{ and } \varepsilon \to 0$$

$$\therefore \quad \lim_{n\to\infty} F_{X_n}(t) = F_X(t) \qquad \forall\ t \in C(F)$$

ie $\quad X_n \xrightarrow{d} X$

<u>Slutsky's Theorem</u> : If $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{P} a$, $Z_n \xrightarrow{P} b$

then $\quad X_n Y_n + Z_n \xrightarrow{d} aX + b$

[X_n, Y_n, Z_n must be defined on a common prob space but X need not be def on the same space]

<u>Proof</u> : <u>Step 1</u> If $U_n - V_n \xrightarrow{P} 0$ and $U_n \xrightarrow{d} U \Rightarrow V_n \xrightarrow{d} U$

<u>Step 2</u> If $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{P} a \rightarrow X_n Y_n \xrightarrow{d} aX$

<u>Step 3</u> If $X_n \xrightarrow{d} X$ and $Z_n \xrightarrow{P} b \rightarrow X_n + Y_n \xrightarrow{d} X + b$

<u>Remark</u> : If $X_1, X_2 \ldots$ are indep r.v with a common distn functn F_0 then $X_n \xrightarrow{d} X_0$ for any random variable X_0 with df F_0.

Property $\quad X_n \xrightarrow{d} a \iff X_n \xrightarrow{P} a \quad [a = \text{const}]$

"$\Leftarrow$" given $X_n \xrightarrow{P} a \implies X_n \xrightarrow{d} a$ (by theorem)

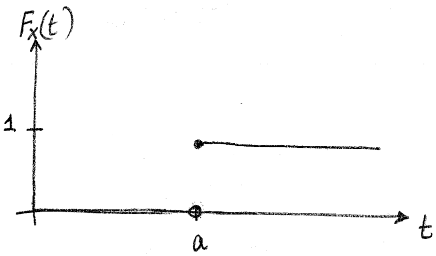
"$\Rightarrow$" given $X_n \xrightarrow{d} a$ to show $X_n \xrightarrow{P} a$ ie $P(|X_n - a| > \varepsilon) \longrightarrow 0$

Consider
$$\begin{aligned} P(|X_n-a|>\varepsilon) &= P([X_n-a>\varepsilon] \cup [X_n-a<-\varepsilon]) \\ &= P(X_n-a>\varepsilon) + P(X_n-a<-\varepsilon) && \{\text{disjoint union} \\ &= P(X_n>\varepsilon+a) + P(X_n<a-\varepsilon) \\ &= [1-P(X_n\le a+\varepsilon)] + P(X_n<a-\varepsilon) \\ &\le [1-F_{X_n}(a+\varepsilon)] + P(X_n \le a-\varepsilon) && \{\because \{X<c\} \subset \{X \le c\} \end{aligned}$$

$$P(|X_n-a|>\varepsilon) \le 1 - F_{X_n}(a+\varepsilon) + F_{X_n}(a-\varepsilon)$$

Now $X(\omega) = a \quad \forall\, \omega$

$$F_X(t) = P(X \le t) = P(\{\omega \in \Omega : X(\omega) \le t\})$$
$$= \begin{cases} P(\Omega) & \text{if } t \ge a \\ P(\phi) & \text{if } t < a \end{cases}$$



$\therefore$ a is the only pt of discontinuity of F_X

ie $a-\varepsilon$ and $a+\varepsilon$ are both $\in C(F_X)$ and since $X_n \xrightarrow{d} X$ $\Big\} \Rightarrow$ $F_{X_n}(a+\varepsilon) \longrightarrow F_X(a+\varepsilon)$, $F_{X_n}(a-\varepsilon) \longrightarrow F_X(a-\varepsilon)$

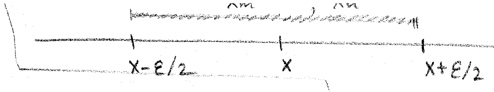
$$\therefore \quad 0 \le \underline{\lim}\, P(|X_n-a|>\varepsilon) \le 1 - F_X(a+\varepsilon) + F_X(a-\varepsilon) = 1-1+0 = 0$$

$$0 \le \overline{\lim}\, P(|X_n-a|>\varepsilon) \le 1 - F_X(a+\varepsilon) + F_X(a-\varepsilon) = 1-1+0 = 0$$

$$\therefore \quad \underline{\lim}\, P(|X_n-a|>\varepsilon) = 0 = \overline{\lim}\, P(|X_n-a|>\varepsilon)$$

$$\Rightarrow \quad P(|X_n-a|>\varepsilon) \longrightarrow 0$$

ie $\quad X_n \xrightarrow{P} a$



so max diff b/w X_n & X_m = length = $2(\varepsilon/2)$

Theorem: $X_n \xrightarrow{\mu} X \iff X_m - X_n \xrightarrow{\mu} 0$

Proof: "$\Rightarrow$" given $X_n \xrightarrow{\mu} X$ to show $\mu(|X_m - X_n| > \varepsilon) \to 0 \;\forall\, \varepsilon > 0$.

$$[|X_m - X| \leq \varepsilon/2] \text{ and } [|X_n - X| \leq \varepsilon/2] \subseteq [|X_m - X_n| \leq \varepsilon]$$

$$\therefore \mu([|X_m - X_n| > \varepsilon]) \leq \mu([|X_m - X| > \varepsilon/2] \cup [|X_n - X| > \varepsilon/2])$$
$$\leq \mu(|X_m - X| > \varepsilon/2) + \mu(|X_n - X| > \varepsilon/2)$$

$$\therefore 0 \leq \lim_{m \wedge n \to \infty} \mu(|X_m - X_n| > \varepsilon) \leq 0$$

$$\therefore X_m - X_n \xrightarrow{\mu} 0$$

"$\Leftarrow$" Given $X_m - X_n \xrightarrow{\mu} 0$ to show $X_n \xrightarrow{\mu} X$

We proceed as follows
(1) Extract a subseq X_{n_k} st $\mu(|X_{n_k} - X_m| > 1/2^k) \leq 1/2^k$
(2) Define a set C st $\mu(C^c) = 0$
(3) Show $\{X_{n_k}\}$ is cauchy on C & hence has a limit on C.
(4) Show $X_n \xrightarrow{\mu} \lim$

(1) Since $\mu(|X_m - X_n| > \varepsilon) \leq \varepsilon \;\forall\, \varepsilon > 0$ & $m, n \geq n_\varepsilon$; we can select n_k such that

$$\mu(|X_{n_k} - X_m| > \varepsilon) \leq \varepsilon \quad \forall\, \varepsilon > 0 \text{ and } m \geq n_k \geq n_\varepsilon$$

ie we can obt a subseq st $\mu(|X_{n_k} - X_m| > 1/2^k) \leq 1/2^k$ $\quad \{\varepsilon \equiv 1/2^k$

(2) Let $A_k \equiv \{|X_{n_k} - X_{n_{k+1}}| > 1/2^k\}$

$$B_m = \bigcup_{k \geq m}^{\infty} A_k \qquad [\text{Notice } B_m \downarrow \text{seq}]$$

$$C = \bigcup_{m=1}^{\infty} B_m^c$$

$$\mu(C^c) = \mu\left(\bigcap_{m=1}^{\infty} B_m\right) = \lim_{m \to \infty} \mu(B_m) = \lim_{m \to \infty} \mu\left(\bigcup_{k \geq m}^{\infty} A_k\right)$$
$$\leq \lim_{m \to \infty} \sum_{k=m}^{\infty} \mu(A_k)$$

$$0 \leq \mathcal{M}(C^c) \leq \lim_{m\to\infty} \sum_{k\geq m}^{\infty} \left(\frac{1}{2^k}\right) = \lim_{m\to\infty} \left(\frac{1}{2}\right)^{m-1} \qquad \left\{ \left(\tfrac{1}{2}\right)^m + \left(\tfrac{1}{2}\right)^{m+1} + \cdots = \left(\tfrac{1}{2}\right)^m * \left[\frac{1}{1-1/2}\right] = \left(\tfrac{1}{2}\right)^{m-1} \right.$$

$$\therefore \mathcal{M}(C^c) = 0$$

(3) Now let $\omega_0 \in C$ be arbitrary $\Rightarrow \omega_0 \in \bigcup_{m=1}^{\infty} B_m^c$

$$\Rightarrow \omega_0 \in B_{m_0}^c \quad \text{for some } m_0$$

$$\Rightarrow \omega_0 \in \bigcap_{k=m_0}^{\infty} A_k^c$$

$$\Rightarrow \omega_0 \in A_k^c \quad \forall\ k \geq m_0$$

$$\Rightarrow \omega_0 \in \left\{ |X_{n_k} - X_{n_{k+1}}| \leq \frac{1}{2^k} \right\} \quad \forall\ k \geq m_0$$

ie $\forall\ k \geq m_0$, $|X_{n_k}(\omega_0) - X_{n_{k+1}}(\omega_0)| \leq \frac{1}{2^k}$

Now $\forall\ m_i > m_j \geq m_0$, $|X_{m_j} - X_{m_i}| = |X_{m_j} - X_{m_{j+1}} + X_{m_{j+1}} - X_{m_{j+2}} + X_{m_{j+2}} \cdots - X_{m_i}|$

$$\leq |X_{m_j} - X_{m_{j+1}}| + |X_{m_{j+1}} - X_{m_{j+2}}| + \cdots + |X_{m_{i-1}} - X_{m_i}| \quad [\ \&\ \omega \in C$$

$$\leq |X_{m_j} - X_{m_{j+1}}| + |X_{m_{j+1}} - X_{m_{j+2}}| + \cdots$$

$$\leq \sum_{k=j}^{\infty} 1/2^k = \left(\frac{1}{2}\right)^{j-1} \quad (\equiv \varepsilon \text{ here})$$

$$\rightarrow |X_{m_j} - X_{m_i}| < \varepsilon \quad \forall\ m_i > m_j \geq m_0 \text{ and } \omega \in C$$

ie $\{X_{n_k}\}$ is cauchy on C $\Rightarrow$ $\lim X_{n_k}$ exists $\equiv X$ on C

ie $\lim X_{n_k} = \begin{cases} X & \forall\ \omega \in C \\ 0 & \text{ow} \end{cases}$ ie $X_{n_k} \xrightarrow{a.e} X$

Now $|X_n - X| \leq |X_n - X_{n_k}| + |X_{n_k} - X|$

$$\mathcal{M}(|X_n - X| > \varepsilon) \leq \underbrace{\mathcal{M}(|X_n - X_{n_k}| > \varepsilon)}_{\downarrow \text{ cauchy }\ 0} + \underbrace{\mathcal{M}(|X_{n_k} - X| > \varepsilon)}_{\downarrow\ 0}$$

$X_{n_k} \xrightarrow{a.e} X$ & $\mathcal{M}(\Omega) < \infty$

$$\therefore 0 \leq \lim_{n\to\infty} \mathcal{M}(|X_n - X| > \varepsilon) \leq 0 \quad \Rightarrow \quad X_n \xrightarrow{\mathcal{M}} X$$

LEBESGUE INTEGRAL

- If $X \geq 0$ is a simple funct^n ie $X = \sum_{i=1}^{n} x_i I_{A_i}$; $\sum_{i=1}^{n} A_i = \Omega$, $x_i \geq 0 \in \mathbb{R}$ then

$$\int X \cdot d\mu \equiv \sum_{i=1}^{n} x_i \mu(A_i)$$

If $X \geq 0$ then $\int X \cdot d\mu \equiv \sup \left\{ \int Y \cdot d\mu \ : \ 0 \leq Y \leq X \text{ and } Y \text{ is a simple function} \right\}$

If X is any arbitrary funct^n $\int X \cdot d\mu \equiv \int X^+ d\mu - \int X^- d\mu$ if at least one is finite

[Note : X must be measurable for the integral to be defined]

- If X is not measurable but $X = Y$ a.e and Y is measurable then

$$\int X \cdot d\mu = \int Y \cdot d\mu$$

- If $\int X \cdot d\mu < \infty$ then $X : (\Omega, \mathcal{A}, \mu) \longrightarrow (\mathbb{R}, \mathcal{B})$ is an integrable function.

<u>Proposition</u> <u>$\int X \cdot d\mu$</u> is <u>well defined</u>.

Proof : To show if $X = \sum_{i=1}^{n} x_i I_{A_i} = \sum_{j=1}^{m} y_j I_{B_j}$ then $\int X \cdot d\mu$ is well defined

Now $X = \sum_{i=1}^{n} x_i I_{A_i} = \sum_{i=1}^{n} x_i I_{A_i} \sum_{j=1}^{m} I_{B_j} = \sum_{i=1}^{n} x_i \sum_{j=1}^{m} I_{A_i B_j}$ $\left\{ \because \sum_{j=1}^{m} B_j = \Omega \right.$

also $X = \sum_{j=1}^{m} y_j I_{B_j} = \sum_{j=1}^{n} y_j I_{B_j} \sum_{i=1}^{m} I_{A_i} = \sum_{j=1}^{m} y_j \sum_{i=1}^{n} I_{A_i B_j}$

$$\therefore \sum_{i=1}^{n} \sum_{j=1}^{m} x_i I_{A_i B_j} = X = \sum_{i=1}^{n} \sum_{j=1}^{m} y_j I_{A_i B_j}$$

So if $A_i B_j \neq \phi \Rightarrow x_i = y_j$

Now $\int X \cdot d\mu = \sum_{i=1}^{n} x_i \, \mu(A_i) = \sum_{i=1}^{n} x_i \sum_{j=1}^{m} \mu(A_i B_j) = \sum_{i=1}^{n}\sum_{j=1}^{m} x_i \, \mu(A_i B_j)$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m} y_j \, \mu(A_i B_j)$$

$$= \int X \cdot d\mu$$

So it is well defined for simple funct^n and hence for the rest.

<u>Proposition</u> <u>Elementary properties of the integral</u> If X and Y are meas and $\int X, \int Y$ are well defin

1) $\int (X+Y)\, d\mu = \int X \cdot d\mu + \int Y \, d\mu$.

2) $\int cX \cdot d\mu = c \int X \cdot d\mu$

3) $X \geq 0 \Rightarrow \int X d\mu \geq 0$.

<u>Proof</u> : (3) <u>Step 1</u> If $X \geq 0$ is a simple function $X = \sum_{i=1}^{n} x_i I_{A_i}$; $x_i \geq 0$ $\sum_{i=1}^{n} A_i = \Omega$

$\int X \cdot d\mu = \sum_{i=1}^{n} x_i \, \mu(A_i) \geq 0$ $\{$since $x_i \geq 0$, $\mu(A_i) \geq 0 \}$

<u>Step 2</u> If $X \geq 0$ then by def $\int X \cdot d\mu = \sup\{ \int Y \cdot d\mu : 0 \leq Y \leq X$, Y is simple func$^n\}$

Since $Y \geq 0$ is a simple funct^n $\rightarrow \int Y \, d\mu \geq 0 \rightarrow \sup\{\int Y \cdot d\mu\} \geq 0$.

(2) <u>Step 1</u> X is a simple ≥ 0 function. $X = \sum_{i=1}^{n} x_i I_{A_i}$ $x_i \geq 0$ & $c > 0$

$cX = \sum_{i=1}^{n} c x_i I_{A_i} = \sum_{i=1}^{n} y_i I_{A_i}$ $\{ y_i = c x_i$

$\int cX d\mu = \sum_{i=1}^{n} y_i \, \mu(A_i) = c \sum_{i=1}^{n} x_i \, \mu(A_i) = c \int X \cdot d\mu$

<u>Step 2</u> $X \geq 0$ and $c \geq 0$ then $cX \geq 0$ also

$c \int X \cdot d\mu = c \cdot \sup\{ \int Y \, d\mu : 0 \leq Y \leq X$ and $Y \geq 0$ is simple $\}$

$= \sup\{ c \int Y \cdot d\mu ; 0 \leq Y \leq X$, Y simple $\}$

$= \sup\{ \int cY \cdot d\mu : 0 \leq cY \leq cX$, Y simple $\}$ $\{ \because$ by step 1 $\}$

$= \int cX \cdot d\mu$

Step 3 : X any arbitrary meas funct^n $\quad X = X^+ - X^-$

Also $\quad cX = cX^+ - cX^- = \begin{cases} Y^+ - Y^- & \text{if } c>0, \; Y^+ = cX^+ \;\&\; Y^- = cX^- \\ -Y^- + Y^+ & \text{if } c<0, \; Y^+ = -cX^-, \; Y^- = -cX^+ \end{cases}$

$$\int cX = \int Y \cdot d\mu = \int Y^+ - \int Y^- = \begin{cases} \int cX^+ - \int cX^- & \text{if } c>0 \\ \int -cX^- - \int -cX^+ & \text{if } c<0 \end{cases}$$

$$\int cX \cdot d\mu = \begin{cases} c\int X^+ - c\int X^- & \text{if } c>0 \\ -c\int X^- + c\int X^+ & \text{if } c<0 \end{cases} \quad [\because \text{ by step 2}]$$

$$= c\left(\int X^+ - \int X^-\right)$$

$$= c\int X \cdot d\mu$$

(1) To show $\int (X+Y)\, d\mu = \int X\, d\mu + \int Y \cdot d\mu$

Step 1 Suppose X & Y are ≥ 0 simple functions

$$X = \sum_{i=1}^{n} x_i I_{A_i} = \sum_{i=1}^{n} x_i I_{A_i} \sum_{j=1}^{m} I_{B_j} = \sum_i^n \sum_j^m x_i I_{A_i B_j}$$

$$Y = \sum_{j=1}^{m} y_j I_{B_j} = \sum_{j=1}^{m} y_j I_{B_j} \sum_{i=1}^{n} I_{A_i} = \sum_i^n \sum_j^m y_j I_{A_i B_j}$$

$$(X+Y) = \sum_i^n \sum_j^m (x_i + y_j) I_{A_i B_j}$$

$$\int X \cdot d\mu + \int Y \cdot d\mu = \sum_{i=1}^{n} x_i \mu(A_i) + \sum_{j=1}^{m} y_j \mu(B_j)$$

$$= \sum_i^n x_i \sum_j^m \mu(A_i B_j) + \sum_j^m y_j \sum_i^n \mu(A_i B_j)$$

$$= \sum_i^n \sum_j^m x_i \mu(A_i B_j) + \sum_i^n \sum_j^m y_j \mu(A_i B_j)$$

$$= \sum_i^n \sum_j^m (x_i + y_j) \mu(A_i B_j)$$

$$= \int (X+Y) \cdot d\mu$$

Step 2 $\quad X \geq 0$, $Y \geq 0$ and X, Y are measurable (using meas. via simple funct^n.....)

$\therefore \exists$ seq of ≥ 0 simple functions $\{X_n\}$ & $\{Y_n\}$ st $X_n \uparrow X$ and $Y_n \uparrow Y$

Also $(X_n + Y_n) \geq 0$ and simple and $(X_n + Y_n) \uparrow (X+Y)$

Now by MCT $\Rightarrow$
$$\begin{cases} \int X_n \cdot d\mu \uparrow \int X \cdot d\mu \\ \int Y_n \cdot d\mu \uparrow \int Y \cdot d\mu \\ \int (X_n + Y_n)\, d\mu \uparrow \int (X+Y) \cdot d\mu \end{cases}$$

Now
$$\int X \cdot d\mu + \int Y \cdot d\mu = \lim_{n\to\infty} \int X_n \, d\mu + \lim_{n\to\infty} \int Y_n \, d\mu$$
$$= \lim_{n\to\infty} \left[\int X_n \, d\mu + \int Y_n \cdot d\mu \right]$$
$$= \lim_{n\to\infty} \left[\int (X_n + Y_n) \cdot d\mu \right] \qquad \text{by step 1 for simple fn}$$
$$= \int (X+Y) \, d\mu$$

Step 3 $\quad X$ and Y are arbitrary meas functions.

$X = X^+ - X^-$ and $Y = Y^+ - Y^-$

$$X + Y = X^+ - X^- + Y^+ - Y^- = (X^+ + Y^+) - (X^- + Y^-)$$

$$\int X \cdot d\mu + \int Y \cdot d\mu = \int X^+ - \int X^- + \int Y^+ - \int Y^-$$
$$= \left(\int X^+ + \int Y^+ \right) - \left(\int X^- + \int Y^- \right)$$
$$= \int (X^+ + Y^+) - \int (X^- + Y^-)$$
$$= \int (X+Y) \, d\mu$$

<u>The Monotone Convergence Theorem</u>. $\{X_n\}$ is an $\uparrow$ seq of measurable funct^n, $X_n \geq 0$ st $X_n \uparrow X$ a.e then $\int X_n \, d\mu \uparrow \int X \cdot d\mu$

Pf (i) Since $\lim_{n\to\infty} X_n = X \quad \forall \omega \in N$, we can redefine the limit on the set N^c with $\mu(N^c)=0$ so we can say $\lim_{n\to\infty} X_n = X$ and $X \geq 0$

Since X_n's are measurable $\rightarrow$ X is also measurable.

(ii) $\{X_n\} \uparrow \geq 0$ seq $\Rightarrow 0 \leq X_n \leq X_{n+1} \leq X_{n+2} \dots\dots$

$\Rightarrow \quad 0 \leq \int X_n \leq \int X_{n+1} \leq \dots\dots$ $\{$ by elementary prop. of integral

ie $\quad a_n \equiv \left(\int X_n \cdot d\mu\right) \uparrow$ and $\geq 0 \quad \Rightarrow \quad a_n$ has a limit $\in [0,\infty]$

Let $\quad a \equiv \lim_{n\to\infty} a_n = \lim_{n\to\infty} \int X_n \cdot d\mu$

(iii) Now $\quad X_n \uparrow X \quad \Rightarrow \quad X_n \leq X \quad \forall n.$

$\Rightarrow \quad \int X_n \, d\mu \leq \int X \cdot d\mu \quad \forall n$

$\Rightarrow \quad \lim_{n\to\infty} \int X_n \cdot d\mu \leq \int X \cdot d\mu \quad \forall n$

$\Rightarrow \quad a \leq \int X \cdot d\mu$

(iv) We need to show $\quad a \geq \int X \cdot d\mu \quad$ ie $\quad \lim_{n\to\infty} \int X_n \, d\mu \geq \int X \cdot d\mu$

We will show $\quad a \geq \theta \int X \cdot d\mu \quad \forall \theta \in [0,1] \Rightarrow a \geq \lim_{\theta\to 1} \theta \int X \cdot d\mu = \int X \cdot d\mu$

Now since $X \geq 0 \Rightarrow \int X \cdot d\mu \equiv \sup\left\{ \int Y \cdot d\mu : 0 \leq Y \leq X \text{ and } Y \text{ is a simple funct}^n \right\}$

$\therefore$ we need to show $\quad \theta \int Y \cdot d\mu \leq a$ $\left[\text{which implies } \sup\left\{\theta \int Y \cdot d\mu\right\} \leq a\right]$ for all $0 \leq Y \leq X$, Y = simple functions

Let $0 \le \theta \le 1$ be arbitrary, fixed and $0 \le y \le X$ where y is a simple funct^n

Define $A_n = \{ \omega \in \Omega \mid \theta y(\omega) \le X_n(\omega) \}$

Since $\{X_n\} \uparrow \Rightarrow \{A_n\} \uparrow$

Also $\bigcup_{n=1}^{\infty} A_n = \Omega$

(a) $A_n \subset \Omega \Rightarrow \bigcup_{n=1}^{\infty} A_n \subset \Omega$

(b) "$\supseteq$" Let $\omega \in \Omega$ and $X(\omega) > 0$

Since $X(\omega) = \lim_{n\to\infty} X_n(\omega) \ge y(\omega) > \theta y(\omega) \Rightarrow \exists\, n_\varepsilon$ st $X_n(\omega) > \theta y(\omega)$ $\forall\, n \ge n_\varepsilon$

$\Rightarrow \omega \in A_n \quad \forall\, n \ge n_\varepsilon$

$\Rightarrow \omega \in \bigcup_{n=1}^{\infty} A_n$

Let $\omega \in \Omega$ and $X(\omega) = 0$

then since $0 \le y(\omega) \le X(\omega) = 0 \Rightarrow y(\omega) = 0$

$\Rightarrow y(\omega) \le X_n(\omega) \quad \forall n \quad \{\because X_n \ge 0$

$\Rightarrow \omega \in A_n \quad \forall n$

$\Rightarrow \omega \in \bigcup_{n=1}^{\infty} A_n$

Now $\theta \int_{A_n} y \, d\mu = \int_{A_n} \theta y \, d\mu \le \int_{A_n} X_n \, d\mu \le \int X_n \, d\mu \le \lim_{n\to\infty} \int X_n \, d\mu \equiv a$

$\left\{ \begin{array}{l} \because X_n \uparrow X \Rightarrow X_n \le X \quad \forall n \\ \Rightarrow \int X_n \le \int X \quad \forall n \end{array} \right.$

$\therefore \theta \int y I_{A_n} \, d\mu \le a$

Now $y = \sum_{j=1}^{m} y_j I_{B_j} \Rightarrow y I_{A_n} = \sum_{j=1}^{m} y_j I_{A_n B_j}$

$\Rightarrow \int y I_{A_n} \, d\mu = \sum_{j=1}^{m} y_j \mu(A_n B_j)$

Now $\{A_n\} \uparrow$ seq $\Rightarrow \{A_n B_j\}_n \uparrow$ seq also $\Rightarrow \lim_{n\to\infty} \mu(A_n B_j) = \mu\left(\bigcup_{n=1}^{\infty} A_n B_j\right)$

$= \mu\left(B_j \bigcup_{n=1}^{\infty} A_n\right) = \mu(B_j)$

$$\therefore \lim_{n\to\infty} \int y I_{A_n} d\mu = \lim_{n\to\infty} \sum_{j=1}^{m} y_j \mu(A_n B_j) = \sum_{j=1}^{m} y_j \lim_{n\to\infty} \mu(A_n B_j)$$

$$= \sum_{j=1}^{m} y_j \mu(B_j)$$

$$= \int y \cdot d\mu$$

$$\therefore \theta \int_{A_n} y \, d\mu \uparrow \theta \int y \cdot d\mu$$

Since $\theta \int_{A_n} y \cdot d\mu \leq a \Rightarrow \lim_{n\to\infty} \theta \int_{A_n} y \cdot d\mu \leq a$

$$\Rightarrow \theta \int y \cdot d\mu \leq a$$

<u>Fatou's lemma</u>. $\int \underline{\lim} \, X_n \, d\mu \leq \underline{\lim} \int X_n \, d\mu$ if $X_n \geq 0^*$ a.e $\forall n$

Proof: $\underline{\lim} \, X_n(\omega) = \lim_{n\to\infty} \left(\inf_{k\geq n} X_k(\omega) \right) = \lim_{n\to\infty} (Y_n)$ $\left\{ Y_n = \inf_{k\geq n} X_k \right.$

$$= \sup_{n\geq 1} (Y_n) \qquad \left\{ \because Y_n \uparrow \right.$$

$$= \sup_{n\geq 1} \left(\inf_{k\geq n} X_k(\omega) \right)$$

$X_n \geq \inf_{k\geq n} X_k \equiv Y_n \ \forall n$. and $\inf_{k\geq n} X_k \uparrow \underline{\lim} \, X_n$

ie $Y_n \uparrow \underline{\lim} \, X_n \underset{\text{by MCT}}{\Longrightarrow} \int Y_n \, d\mu \uparrow \int \underline{\lim} \, X_n$ & $\lim \int Y_n = \int \lim Y_n$

Now $\int \underline{\lim} \, X_n = \int \lim Y_n = \lim \int Y_n = \underline{\lim} \int Y_n \leq \underline{\lim} \int X_n$

$\left\{ \because Y_n \leq X_n \right.$

The Dominated Convergence Theorem

$\{X_n\}$ is a seq of measurable funct$^{\underline{n}}$

If $|X_n| \le Y$ a.e and $Y \in \mathcal{L}_1 \equiv \{g : \int |g| \, d\mu < \infty\}$ and if the foll hold

(a) $X_n \xrightarrow{a.e} X$ OR (b) $X_n \xrightarrow{\mu} X$

then $\int |X_n - X| \cdot d\mu \longrightarrow 0$ ie $X_n \xrightarrow[\mathcal{L}_1]{} X$

Note $\sup_{n \ge 1} |X_n| \equiv Y$ is suitable so long as $Y \in \mathcal{L}_1$

Proof (a) $X_n \xrightarrow{a.e} X$ (given) to show $\int |X_n - X| \, d\mu \to 0$

Let $Z_n \equiv |X_n - X| \ge 0$

Also $Z_n \xrightarrow{a.e} 0 \Rightarrow \underline{\lim}\, Z_n = 0 \Rightarrow \int \underline{\lim}\, Z_n = 0$

We will show $0 \le \underline{\lim} \int Z_n \, d\mu \le \overline{\lim} \int Z_n \, d\mu \le 0$

Now $0 = \int \underline{\lim}\, Z_n \, d\mu \le \underline{\lim} \int Z_n \, d\mu$ (by Fatou's lemma)

$Z_n = |X_n - X| \le |X_n| + |X| \le Y + Y$ $\left\{ \because \begin{array}{l} |X_n| \le Y \quad \forall n \\ X_n \to X \quad a.e \end{array} \right.$

$\therefore Z_n \le 2Y$

$\therefore 2Y - Z_n \ge 0$

$\int \underline{\lim}\, (2Y - Z_n) \le \underline{\lim} \int (2Y - Z_n)$ $\left\{ \begin{array}{l} \text{by applying Fatou's lemma to} \\ (2Y - Z_n) \ge 0 \end{array} \right.$

$\Rightarrow \int (2Y - \underline{\lim}\, Z_n) \le \underline{\lim} \left(\int 2Y + \int -Z_n \right)$

$\int 2Y \le \int 2Y + \underline{\lim} \left(-\int Z_n \right)$ $\{\because \underline{\lim}\, Z_n = 0$

$0 \le -\overline{\lim} \left(\int Z_n \right)$ $\left\{ \begin{array}{l} Y \in \mathcal{L}_1 \text{ so you can cancell} \\ \underline{\lim}(a_n) = -\overline{\lim}(-a_n) \end{array} \right.$

$0 \ge \overline{\lim} \left(\int Z_n \right)$

$\therefore \; 0 \leq \underline{\lim} \int Z_n \leq \overline{\lim} \int Z_n \leq 0 \Rightarrow \int Z_n \longrightarrow 0$ ie $\int |X_n - X| \longrightarrow 0$

(b) Given $X_n \xrightarrow{\mu} X$ to show $\int |X_n - X| \, d\mu \longrightarrow 0$

Let $Z_n \equiv |X_n - X| \geq 0 \Rightarrow \overline{\lim} \; Z_n \geq 0$ and also $\int Z_n \geq 0$

Also $X_n \xrightarrow{\mu} X \Rightarrow Z_n \xrightarrow{\mu} 0$

$\underline{\lim} \left(\int Z_n \, d\mu \right) \leq \overline{\lim} \left(\int Z_n \, d\mu \right)$ always.

$Z_n \geq 0 \Rightarrow \int Z_n \geq 0 \Rightarrow \underline{\lim} \left(\int Z_n \right) \geq 0$

$\therefore \; 0 \leq \underline{\lim} \int Z_n \leq \overline{\lim} \int Z_n \equiv a$

We need to show $a = 0$.

Now since $\overline{\lim} \left(\int Z_n \right) = a$ we can find a subseq n' st $\lim\limits_{n' \to \infty} \int Z_{n'} = a$

Since $Z_n \xrightarrow{\mu} 0 \Rightarrow Z_{n'} \xrightarrow{\mu} 0$ for subseq n' (and also for any other subseq)

If $Z_{n'} \xrightarrow{\mu} 0 \quad \exists$ a subseq n'' st $Z_{n''} \xrightarrow{a.e} 0$ [by theorem 2.3.1]

So by part ⓐ $\Rightarrow \int Z_{n''} \longrightarrow 0$ — (*)

But $\{Z_{n''}\}$ is a subseq of $\{Z_{n'}\} \Rightarrow \int Z_{n''} \longrightarrow a$ (also) — (**)

$\therefore$ (*) & (**) $\Rightarrow a = 0$

ie $\int Z_n \longrightarrow 0$ as $n \to \infty$

Corollary : $\int |X_n - X| \, d\mu \longrightarrow 0 \Rightarrow \int X_n \longrightarrow \int X \, d\mu$

$0 \leq \left| \int X_n \cdot d\mu - \int X \cdot d\mu \right| \leq \int |X_n - X| \cdot d\mu \longrightarrow 0$

$\therefore \lim\limits_{n \to \infty} \left(\int X_n d\mu - \int X \, d\mu \right) = 0 \Rightarrow \int X_n \, d\mu \longrightarrow \int X \cdot d\mu$

Corollary $\int |X_n - X|\, d\mu \longrightarrow 0 \quad \Rightarrow \quad \sup_A \left| \int_A X_n\, d\mu - \int_A X\, d\mu \right| \longrightarrow 0$

Proof

$$\left| \int_A X_n \cdot d\mu - \int_A X \cdot d\mu \right| = \left| \int_A (X_n - X) \cdot d\mu \right| = \left| \int (X_n - X) I_A \, d\mu \right|$$

$$\leq \int |X_n - X| \, I_A \, d\mu$$

$$\leq \int |X_n - X| \cdot d\mu \qquad \forall A$$

$\therefore$

$$\left| \int_A X_n\, d\mu - \int_A X \cdot d\mu \right| \longrightarrow 0 \qquad \forall A \qquad \text{as } n \to \infty$$

$\therefore$

$$\sup_A \left| \int_A X_n\, d\mu - \int_A X \cdot d\mu \right| \longrightarrow 0 \qquad \text{as } n \to \infty$$

Theorem If $X_n \geq 0$ a.e then $\int \sum_{n=1}^{\infty} X_n \cdot d\mu = \sum_{n=1}^{\infty} \int X \cdot d\mu$

ie $$\int \lim_{n \to \infty} \sum_{k=1}^{n} X_k \, d\mu = \lim_{n \to \infty} \sum_{k=1}^{n} \int X_k \, d\mu$$

Proof

Let $Z_n = \sum_{k=1}^{n} X_k$

Since $X_n \geq 0 \Longrightarrow Z_n \uparrow$ and ≥ 0

$$\lim_{n \to \infty} Z_n = \lim_{n \to \infty} \sum_{k=1}^{n} X_k = \sum_{k=1}^{\infty} X_k \equiv Z$$

$\therefore$ $\{Z_n\} \uparrow Z$ a.e and $Z_n \geq 0$ so by applying the MCT we get

$$\int Z_n \, d\mu \uparrow \int Z \, d\mu$$

ie $$\lim_{n \to \infty} \int Z_n \, d\mu = \int Z \, d\mu$$

ie $$\lim_{n \to \infty} \left(\int \sum_{k=1}^{n} X_k \right) = \int \lim_{n \to \infty} \sum_{k=1}^{n} X_k$$

ie $\lim_{n\to\infty}\left(\sum_{k=1}^{n}\int X_k \, d\mu\right) = \int \lim_{n\to\infty}\sum_{k=1}^{n} X_k \, d\mu$ (by linearity of integrals in the finite case

ie $\sum_{k=1}^{\infty}\int X_k \, d\mu = \int \sum_{k=1}^{\infty} X_k \, d\mu$

Theorem: The absolute continuity of the integral

Let $X \in \mathcal{L}_1 \equiv \{g : \int |g| \, d\mu < \infty\}$ then if $\mu(A) \to 0$ for any set $A \Rightarrow \int_A |X| \, d\mu \to 0$

Proof: $\int |X| \, d\mu = \int |X| \, I_{[|X|>n]} + \int |X| \, I_{[|X|\leq n]}$ for $n \in N$

As $n \to \infty$, $\{\omega \in \Omega : |X(\omega)| \leq n\} \uparrow \Omega$

$\Rightarrow \quad I_{[|X|\leq n]} \uparrow I_{\Omega}$ as $n \to \infty$

$\Rightarrow \quad |X| \, I_{[|X|\leq n]} \uparrow |X|$ and $|X| \, I_{[|X|\leq n]} \geq 0$

$\Rightarrow \quad \int |X| \, I_{[|X|\leq n]} \, d\mu \uparrow \int |X| \, d\mu$ {by MCT

Now $\lim_{n\to\infty}\int |X| \, d\mu = \lim_{n\to\infty}\int |X| \, I_{[|X|\leq n]} \, d\mu + \lim_{n\to\infty}\int |X| \, I_{[|X|>n]} \, d\mu$

$\therefore \quad \int |X| \, d\mu = \int |X| \, d\mu + \lim_{n\to\infty}\int |X| \, I_{[|X|>n]} \, d\mu$

$\therefore \quad 0 = \lim_{n\to\infty}\int |X| \, I_{[|X|>n]}$

$\therefore \forall \varepsilon > 0 \ \exists n_\varepsilon$ st $\int |X| \, I_{[|X|>n]} \, d\mu \leq \frac{\varepsilon}{2} \quad \forall n \geq n_\varepsilon$

Let A be any arbitrary set with $\mu(A) \to 0$

w.l.g let $\mu(A) \leq \frac{\varepsilon}{2n_\varepsilon}$

$$\int_A |x|\, d\mu = \int_A |x|\, I_{[|x| \le n_\varepsilon]}\, d\mu + \int_A |x|\, I_{[|x| > n_\varepsilon]}\, d\mu$$

$$\le \int_A |x|\, I_{[|x| \le n_\varepsilon]}\, d\mu + \int |x|\, I_{[|x| > n_\varepsilon]}$$

$$\le \int |x|\, I_{[|x| \le n_\varepsilon] \cap A}\, d\mu + \frac{\varepsilon}{2} \qquad \forall\, \varepsilon > 0 \text{ and } n \ge n_\varepsilon$$

$$\le \int_A n_\varepsilon\, d\mu + \frac{\varepsilon}{2}$$

$$= n_\varepsilon\, \mu(A) + \frac{\varepsilon}{2}$$

$$\le n_\varepsilon \left(\frac{\varepsilon}{2 n_\varepsilon}\right) + \frac{\varepsilon}{2}$$

$$= \varepsilon$$

$$\therefore \int_A |x|\, d\mu \le \varepsilon \quad \forall\, \varepsilon > 0 \implies \int_A |x|\, d\mu \to 0$$

<u>*Theorem of the unconscious statistician*</u> :

If $X : (\Omega, \mathcal{A}, \mu) \to (\Omega', \mathcal{A}')$ and $g : (\Omega', \mathcal{A}') \to (\bar{\mathbb{R}}, \bar{\mathcal{B}})$ are measurable funct^{n} $\mu_X(A') \equiv \mu(X^{-1}(A'))$ for $A' \in \mathcal{A}'$ is the induced meas on $(\Omega', \mathcal{A}')$ then

(1) μ_X determines $\mu_{g(x)}$ on $(\bar{\mathbb{R}}, \bar{\mathcal{B}})$ ie $\mu_{g(x)}(B) = \mu_X(g^{-1}(B))$

(2) $$\int_{X^{-1}(A')} g(x)\, d\mu = \int_{A'} g\, d\mu_X \quad \text{for } A' \in \mathcal{A}'$$

$$\begin{tikzcd}
(\Omega, \mathcal{A}, \mu) \arrow[r, "X"] \arrow[dr, "g(x)"'] & (\Omega', \mathcal{A}', \mu_X) \arrow[d, "g"] \\
& (\bar{\mathbb{R}}, \bar{\mathcal{B}}, \mu_{g(x)})
\end{tikzcd}$$

<u>Remark</u> : $X : (\Omega, \mathcal{A}, P) \to (\mathbb{R}, \mathcal{B}, P_X)$ and $g : (\mathbb{R}, \mathcal{B}, P_X) \to (\mathbb{R}, \mathcal{B}, P_{g(x)})$

$$\int g(x)\, dP = \int g\, dP_X = \int g\, dF_X \qquad \text{where } P_X((-\infty, x]) = F_X(x)$$

$$\begin{tikzcd}
(\Omega, \mathcal{A}, P) \arrow[r, "X"] \arrow[dr, "g(x)"'] & (\bar{\mathbb{R}}, \bar{\mathcal{B}}, P_X) \arrow[d, "g"] \\
& (\bar{\mathbb{R}}, \bar{\mathcal{B}}, P_{g(x)})
\end{tikzcd}$$

$\therefore$ given a transf of X namely 'g' all we need is the form of $g(\cdot)$ and $P_X = F_X$ to find expectation (we do not need the dist^{n} $P_{g(x)}$)

Property : $X \geq 0$ and $\int X \cdot d\mu = 0 \;\rightarrow\; \mu([X>0]) = 0$ [ie $X = 0$ a.e]

Proof (by contradiction) Suppose $\mu([X>0]) > 0$

$$[X>0] = \bigcup_{n=1}^{\infty} \{\omega \in \Omega : X(\omega) > 1/n\} \equiv \bigcup_{n=1}^{\infty} A_n$$

$$0 < \mu[X>0] = \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n)$$

{ since $\{A_n\} \uparrow$ seq by monotone prop of meas

$$\therefore \lim_{n\to\infty} \mu(A_n) > 0$$

$$\Rightarrow \exists\, n_0 \text{ st } \mu(A_{n_0}) > 0$$

Now
$$\int X \cdot d\mu \geq \int_{A_{n_0}} X \cdot d\mu$$

$$\geq \int_{A_{n_0}} (1/n_0)\, d\mu \qquad \{\because A_{n_0} = \{\omega : X(\omega) > 1/n_0\}\}$$

$$= \frac{1}{n_0} \mu(A_{n_0})$$

$$> 0$$

we have a contradiction and our supposition is wrong

Property

$$\int_A X \cdot d\mu = 0 \quad \forall A \in \mathcal{A} \;\Rightarrow\; X = 0 \text{ a.e} \qquad \{ X : (\Omega, \mathcal{A}, \mu) \to (\mathbb{R}, \mathcal{B})$$

$$\int_A X \cdot d\mu \geq 0 \quad \forall A \in \mathcal{A} \;\Rightarrow\; X \geq 0 \text{ a.e.}$$

Proof : (i) given $\int_A X \cdot d\mu = 0 \quad \forall A \in \mathcal{A}$

let $A = [X>0] = \{\omega \in \Omega : X(\omega) > 0\} = X^{-1}([0,\infty)) \in \mathcal{A}$ ($\because$ X is $\mathcal{B}$-$\mathcal{A}$ meas.

$$\therefore \int_A X \cdot d\mu = 0 \;\rightarrow\; \int X\, I_{[X>0]}\, d\mu = 0$$

$$\rightarrow \int X^+ \cdot d\mu = 0 \qquad \left(\because \; X^+ = \begin{cases} X & \text{if } X>0 \\ 0 & \text{ow} \end{cases}\right)$$

$$\rightarrow X^+ = 0 \text{ a.e} \qquad \left(\text{since } X^+ \geq 0 \text{ always } \& \int X^+ d\mu = 0\right.$$

Now let $A = [X<0] = \{\omega \in \Omega : X(\omega) < 0\} = X^{-1}((-\infty, 0]) \in \mathcal{A}$ $(\because X \text{ is } \mathcal{B}\text{-}\mathcal{A} \text{ meas})$

$$\therefore \int_A X \cdot d\mu = 0 \Rightarrow \int X \, I_{[X<0]} \cdot d\mu = 0$$

$$\Rightarrow \int -X^- \cdot d\mu = 0 \qquad \left\{ \because X^- = \begin{cases} -X & \text{if } X<0 \\ 0 & \text{ow} \end{cases} \right.$$

$$\Rightarrow \int X^- \cdot d\mu = 0$$

$$\Rightarrow X^- = 0 \text{ a.e} \qquad (\because X^- \geq 0 \text{ always } \& \int X^- d\mu = 0)$$

$\therefore$ we have $\mu([X^- \neq 0]) = 0 = \mu([X^+ \neq 0])$

Now $\mu([X \neq 0]) = \mu(\{\omega \in \Omega : X(\omega) \neq 0\})$

$$\leq \mu([X^+ \neq 0] \cup [X^- \neq 0])$$

$$\leq \mu([X^+ \neq 0]) + \mu([X^- \neq 0])$$

$$0 \leq \mu([X \neq 0]) \leq 0 \Rightarrow \mu([X \neq 0]) = 0 \quad \text{ie} \quad X = 0 \text{ a.e}$$

(ii) given $\int_A X \cdot d\mu \geq 0$ to show $X \geq 0$ a.e

To show $X \geq 0$ ae $\Leftrightarrow X^+ - X^- \geq 0$ a.e

$$\Leftrightarrow \mu([X^+ - X^- < 0]) = 0$$

$$\Leftrightarrow \mu([X^+ < X^-]) = 0$$

$$\Leftrightarrow \mu([X \, I_{[X>0]} < -X \, I_{[X<0]}]) = 0$$

$$\Leftrightarrow \mu([X<0]) = 0 \quad \text{ie} \quad X^- = 0 \text{ a.e}$$

Let $A = [X<0] = \{\omega \in \Omega : X(\omega) < 0\} = X^{-1}((-\infty, 0]) \in \mathcal{A}$

$$\int_A X \cdot d\mu \geq 0 \Rightarrow \int X \, I_{[X<0]} \, d\mu \geq 0$$

$$\Rightarrow \int -X^- \, d\mu \geq 0$$

$$\Rightarrow \int X^- \cdot d\mu \leq 0 \qquad \text{—(*)}$$

But $X^- \geq 0$ always $\Rightarrow \int X^- \cdot d\mu \geq 0$ —(**)

$\therefore$ (*) & (**) $\Rightarrow \int X^- d\mu = 0 \Rightarrow X^- = 0$ a.e $\quad (\because X^- \geq 0 \ \& \int X^- d\mu = 0)$

$\therefore X = X^+$ a.e $\Rightarrow X \geq 0$ a.e.

Property: $(\Omega, \mathcal{A}, \mu)$ is a meas space. $\mathcal{A}_0$ is a sub σ-field of $\mathcal{A}$ $[\mathcal{A}_0 \subset \mathcal{A}$ & $\mathcal{A}_0$ is σ-field$]$ 13

$\mu_0 \equiv \mu|_{\mathcal{A}_0}$ ie $\mu_0(A) = \mu(A) \quad \forall \ A \in \mathcal{A}_0$

then $\int X \cdot d\mu = \int X \cdot d\mu_0$ for any $\mathcal{B}$-$\mathcal{A}_0$ meas function X

Proof: X is a $\mathcal{B}$-$\mathcal{A}_0$ meas funct^n $\rightarrow X^{-1}(\mathcal{B}) \subset \mathcal{A}_0$

$\rightarrow X^{-1}(\mathcal{B}) \subset \mathcal{A} \quad (\because \mathcal{A}_0 \subset \mathcal{A})$

$\therefore$ X is also $\mathcal{B}$-$\mathcal{A}$ measurable

Step 1: X is a simple funct^n ≥ 0, ie $X = \sum_{i=1}^{n} x_i I_{A_i}$ where $A_i \in \mathcal{A}_0$ & $\sum_{i=1}^{n} A_i = \Omega$

$$\int X \cdot d\mu = \sum_{i=1}^{n} x_i \mu(A_i) = \sum_{i=1}^{n} x_i \mu_0(A_i) = \int X \cdot d\mu_0$$

Step 2: $X \geq 0$ and X is $\mathcal{B}$-$\mathcal{A}_0$ meas.

$\therefore \exists$ a seq $\{X_n\} \uparrow \geq 0$ and X_n are simple ≥ 0 funct^n st $X_n \uparrow X$

$$\int X \cdot d\mu = \lim_{n \to \infty} \int X_n \, d\mu \qquad (\because \text{by MCT})$$

$$= \lim_{n \to \infty} \int X_n \, d\mu_0 \qquad (\text{by step 1})$$

$$= \int X \cdot d\mu_0 \qquad (\text{by MCT})$$

step 3: X is any arbitrary $\mathcal{B}$-$\mathcal{A}_0$ meas funct^n.

$X = X^+ - X^-$

$$\int X \cdot d\mu = \int X^+ d\mu - \int X^- d\mu \qquad (\text{by definition})$$

$$= \int X^+ d\mu_0 - \int X^- d\mu_0 \qquad (\text{by step 2 since } X^+ \geq 0 \text{ and } X^- \geq 0)$$

$$= \int X \, d\mu_0 \qquad (\text{by definition})$$

so long as one of $\int X^+ d\mu$ or $\int X^- d\mu < \infty$

Property $\quad m_r^{s-t}\, m_t^{r-s} \geqslant m_s^{r-t} \qquad r \geqslant s \geqslant t \geqslant 0$ where $m_r = E(|X|^r) = \int |X|^r d\mu$

Proof $\quad$ To show $[E(|X|^r)]^{s-t}\,[E(|X|^t)]^{r-s} \geq [E(|X|^s)]^{r-t}$

$$\Leftrightarrow \quad [E(|X|^r)]^{\frac{s-t}{r-t}}\,[E(|X|^t)]^{\frac{r-s}{r-t}} \geqslant [E(|X|^s)]$$

$$\leftrightarrow \quad \{E(|X|^r)\}^{\frac{1}{\left(\frac{r-t}{s-t}\right)}}\,\{E(|X|^t)\}^{\frac{1}{\left(\frac{r-t}{r-s}\right)}} \geqslant E(|X|^s)$$

$$\Leftrightarrow \quad \left\{E\left[|X|^{r\left(\frac{s-t}{r-t}\right)\left(\frac{r-t}{s-t}\right)}\right]\right\}^{\frac{1}{\left(\frac{r-t}{s-t}\right)}}\left\{E\left[|X|^{t\left(\frac{r-s}{r-t}\right)\left(\frac{r-t}{r-s}\right)}\right]\right\}^{\frac{1}{\left(\frac{r-t}{r-s}\right)}} \geqslant E(|X|^s)$$

$$\leftrightarrow \quad \left\{E\left(|X|^{\frac{rs-rt}{r-t}}\right)^{\left(\frac{r-t}{s-t}\right)}\right\}^{\frac{1}{\left(\frac{r-t}{s-t}\right)}}\left\{E\left(|X|^{\frac{rt-st}{r-t}}\right)^{\left(\frac{r-t}{r-s}\right)}\right\}^{\frac{1}{\left(\frac{r-t}{r-s}\right)}} \geqslant E(|X|^s) \qquad \text{—}(*)$$

Now Holders inequality says $\quad E\,|MN| \leqslant \{E(|M|^p)\}^{1/p}\{E(|N|^q)\}^{1/q}$

where $\frac{1}{p}+\frac{1}{q} = 1$ and $p > 1$

Let
$$\left.\begin{aligned} M &\equiv |X|^{\frac{rs-rt}{r-t}} \\ N &\equiv |X|^{\frac{rt-st}{r-t}} \\ p &= \frac{r-t}{s-t} \\ q &= \frac{r-t}{r-s} \end{aligned}\right\} \Rightarrow \quad \begin{aligned} &|MN| = |X|^{\frac{rs-rt+rt-st}{r-t}} = |X|^{\frac{(r-t)s}{r-t}} = |X|^s \\ &r \geqslant s \Rightarrow \frac{r-t}{s-t} \geqslant 1 \\ &\frac{1}{p}+\frac{1}{q} = \left(\frac{r-s}{r-t}\right)+\left(\frac{s-t}{r-t}\right) = \frac{r-t}{r-t} = 1 \end{aligned}$$

$\therefore$ from Hölder's inequality we get $(*)$

ie $\quad [E(|X|^s)]^{r-t} \geqslant [E(|X|^r)]^{s-t}\,[E(|X|^t)]^{r-s} \qquad r \geqslant s \geqslant t$

In particular if $r=4$, $s=2$, $t=1$ we get

$$m_4^1\, m_1^2 \geqslant m_2^3$$

This is called Littlewood's Ineq

Example : $\varepsilon_1, \varepsilon_2 \dots$ are indep r.v st $P(\varepsilon_i = -1) = \frac{1}{2} = P(\varepsilon_i = 1) \quad \forall i$

Show $$A\left(\sum_{i=1}^n a_i^2\right)^{1/2} \leq E\left[\left|\sum_{i=1}^n a_i \varepsilon_i\right|\right] \leq B\left(\sum_{i=1}^n a_i^2\right)^{1/2}$$

Proof

Let $X = \sum_{i=1}^n a_i \varepsilon_i$

$$X^2 = \sum_{i=1}^n a_i^2 \varepsilon_i^2 + 2\sum_{i<j}^n a_i a_j \varepsilon_i \varepsilon_j$$

$$E(|X|^2) = E(X^2) = \sum_{i=1}^n a_i^2 E(\varepsilon_i^2) + 2\sum_{i<j} a_i a_j E(\varepsilon_i)E(\varepsilon_j)$$

$$= \sum_{i=1}^n a_i^2 \qquad \left\{ \because \begin{array}{l} E(\varepsilon_i) = 0 \quad \forall i \\ E(\varepsilon_i^2) = 1 \end{array} \right.$$

Liapunov's inequality says $[E(|X|^r)]^{1/r}$ is $\uparrow$ in r (if $\mu(\Omega) < \infty$)

$$\Rightarrow \quad E(|X|) \leq \{E(|X|^2)\}^{1/2}$$

$$\Rightarrow \quad E\left(\left|\sum_{i=1}^n a_i \varepsilon_i\right|\right) \leq \left(\sum_{i=1}^n a_i^2\right)^{1/2} \qquad \text{—}(*)$$

Now we also have $E(|X|^4)\,[E(|X|)]^2 \geq [E(|X|^2)]^3 \qquad \{ m_1^2\, m_4^1 \geq m_2^3$

$$\Leftrightarrow \quad [E(|X|)] \geq \frac{[E(|X|^2)]^{3/2}}{[E(|X|^4)]^{1/2}}$$

$$\Leftrightarrow \quad E\left(\left|\sum a_i \varepsilon_i\right|\right) \geq \frac{\left(\sum_{i=1}^n a_i^2\right)^{3/2}}{[E(|X|^4)]^{1/2}}$$

$\therefore$ we now need to show $$\frac{1}{[E(|X|^4)]^{1/2}} \geq \frac{1}{D\left(\sum_{i=1}^n a_i^2\right)}$$

We need to show $$D^2\left(\sum_{i=1}^n a_i^2\right)^2 \geq E(|X|^4)$$

Now $$X^4 = \sum_i^n \sum_j^n \sum_k^n \sum_l^n a_i a_j a_k a_l \, \varepsilon_i \varepsilon_j \varepsilon_k \varepsilon_l$$

$$E(|X|^4) = E(X^4) = \sum_i^n \sum_j^n \sum_k^n \sum_l^n a_i a_j a_k a_l \, E(\varepsilon_i \varepsilon_j \varepsilon_k \varepsilon_l)$$

Now $E(\varepsilon_i \varepsilon_j \varepsilon_k \varepsilon_l)$ is non-zero when $i=j=k=l$ (all subscripts are equal) or when pairs of subscripts are equal (eg $i=j \neq k=l$) which can happen in $\binom{4}{2}=6$ ways.

$$\therefore\; E(|X|^4) = \sum_{i=1}^{n} a_i^4 + 6 \sum_{i<k}^{n} a_i^2 a_k^2$$

$$\leqslant 3 \sum_{i=1}^{n} a_i^4 + 6 \sum_{i<k} a_i^2 a_k^2$$

$$= 3\left(\sum_{i=1}^{n} a_i^4 + 2 \sum_{i<k} a_i^2 a_k^2\right)$$

$$= (\sqrt{3})^2 \left(\sum_{i=1}^{n} a_i^2\right)^2$$

$$\therefore\; E(|X|^4) \leqslant (\sqrt{3})^2 \left(\sum_{i=1}^{n} a_i^2\right)^2$$

$$\therefore\; E\left(\left|\sum_{i=1}^{n} a_i \varepsilon_i\right|\right) \geqslant \frac{(\sum a_i^2)^{3/2}}{[E(|X|^4)]^{1/2}}$$

$$\geqslant \frac{1}{\sqrt{3}} \frac{(\sum a_i^2)^{3/2}}{(\sum a_i^2)^1}$$

$$= \frac{1}{\sqrt{3}} \left(\sum_{i=1}^{n} a_i^2\right)^{1/2}$$

$$\therefore\; \frac{1}{\sqrt{3}} \left(\sum_{i=1}^{n} a_i^2\right)^{1/2} \leqslant E\left(\left|\sum_{i=1}^{n} a_i \varepsilon_i\right|\right) \leqslant \left(\sum_{i=1}^{n} a_i^2\right)^{1/2}$$

DEF: $(\mathbb{R}, \hat{\mathcal{B}}_\mu, \mu)$ is the L-S measure space, g is $\hat{\mathcal{B}}_\mu$ meas funct^n then

$\int g\, d\mu$ is called the L-S integral

$\because$ of the correspondence theorem b/w μ & F $\Rightarrow$ $\int g\, d\mu = \int g\, dF$

$$\int_a^b g\, d\mu = \int_a^b g\, dF = \int_{[a,b]} g\, dF = \int g\, I_{[a,b]}\, dF$$

__Theorem:__ For continuous funct^n the L-S integral and the Reimann Stieljes integral coincide

$$\int_a^b g\, d\mu = \int_a^b g\, dF \quad (\text{R-S integral})$$

__L_p-Space__ $\quad L_p = \{ X : \int |X|^p < \infty \}$ for $p > 0$.

__Relationship b/w diff L_p spaces__ $\quad L_s \subset L_r \quad 0 < r < s$ if $\mu(\Omega) < \infty$

__L_r norm:__ $\quad \|X\|_r = \left(\int |X|^r\right)^{1/r} = [E(|X|^r)]^{1/r}$

__Minowski's Ineq__ $\quad \{E(|X+Y|^r)\}^{1/r} \leq \{E(|X|^r)\}^{1/r} + \{E(|Y|^r)\}^{1/r} \quad \forall r \geq 1$

__C_r- Inequality__ $\quad E(|X+Y|^r) \leq 2^{r-1} E(|X|^r) + 2^{r-1} E(|Y|^r) \quad ; \quad r \geq 1$

$\leq E(|X|^r) + E(|Y|^r) \quad ; \quad 0 \leq r < 1$

__Hölders Ineq__ $\quad E(|XY|) \leq \{E(|X|^r)\}^{1/r} \{E(|X|^s)\}^{1/s} \quad ; \quad r > 1,\ s > 0,\ \frac{1}{r} + \frac{1}{s} = 1$

__Cauchy Schwarz's Ineq__ $\quad E(|XY|) \leq \{E(|X|^2)\}^{1/2} \{E(|Y|^2)\}^{1/2}$

__ungs Inequality__ $\quad |ab| \leq \frac{|a|^r}{r} + \frac{|b|^s}{s} \quad ; \quad r, s > 0$ & $\frac{1}{r} + \frac{1}{s} = 1$

__Liapunov's inequality__ $\quad \|X\|_r = \{E(|X|^r)\}^{1/r}$ is $\uparrow$ in r if $\mu(\Omega) < \infty \quad \forall r > 0$

Basic Inequality If $g \geq 0 \uparrow$ on $[0,\infty)$ and X is any meas funct^n

$$\mu(|X| \geq \lambda) \leq \frac{E(g(X))}{g(\lambda)} \qquad \forall \lambda > 0$$

Markov Ineq

$$\mu(|X| \geq \lambda) \leq \frac{E(|X|^r)}{\lambda^r} \qquad \forall \lambda > 0, \; r \geq 1$$

Chebyshev Ineq

$$\mu(|X-\mu| \geq \lambda) \leq \frac{V(X)}{\lambda^2} \qquad \forall \lambda > 0 \; \} \; \mu = E(X)$$

Jensen's Ineq g is a convex funct^n on (a,b) & $\mu((a,b)) = \mu(\Omega) = 1$

then $g[E(X)] \leq E[g(X)]$ if $a < E(X) < b$