1. State and prove Markov's Inequality and Chebyshev's Inequality.

2. Suppose that $\mu(\Omega) < \infty$ and g is continuous. } prop of consistent estimators
Then $X_n \to_\mu X$ implies $g(X_n) \to_\mu g(X)$.

3.
(a) A fair coin is tossed independently n times. Let Sn be the number of heads obtained. Use the Chebyshev's Inequality to find a lower bound of the probability that Sn/n differs from ½ by less than 0.1 when a)n=100 b)n=10,000 c) n=100,000.
(b) In the same setting as above, demonstrate that
$$\lim P(\{ \mid S_n/n - ½ \mid < \varepsilon \}) = 1 \text{ as } n \to \infty \quad \text{for } \varepsilon > 0.$$

4. Prove the Liapunov's Inequality using Jensen's Inequality
(Note: In the textbook they have proved it using Holder's Inequality)

5. (a)Show that if a random variable X is such that $E(|X|^n) \to 0$ as $n \to \infty$, then X is bounded a.s.
(b)Does there exist a random variable X for which $E(|X|^n) = 1/n$ for all $n \geq 2$
(PhD Qualifying Aug 2002)

6. Suppose $E(X) = 0$ and $Var(X) = s^2$.
Prove $P(X \geq a) \leq s^2/(s^2 + a^2)$, $a > 0$.
(Masters Comprehensive Jan 2000).

7. Compute the limit and justify the calculation (Note: For full credit mention clearly the measure space, the sigma algebra, the measure and all the standard results/theorems done in class which you may want to use). Application of DCT

$$\lim \int_{[0,1]} (1+n(x^2))/((1+x^2)^n)dx \text{ as } n \to \infty.$$

8. If $X_n \to_\mu a$,
$Y_n \to_d Z$ and
$Z_n \to_\mu b$.
where a ,b are real constants and Z is a rv.
Prove that $\sqrt{X_n} * Y_n + Z_n^3 \to_d \sqrt{a} * Z + b^3$
(Hint: Slutsky + Qn 2)

NOTE

If $g(X_n) \to g(X)$ $\not\Rightarrow$ $X_n \to X$ $\Rightarrow$ g is continuous

$$P(|X_n - \mu| \geq \lambda) \leq \frac{Var(X_n)}{\lambda^2}.$$

where $\{X_n\}$ all have mean $= \mu$

If $\frac{Var(X_n)}{\lambda^2} \longrightarrow 0$ then $X_n \xrightarrow{P} \mu$ (mean)

$$y = \frac{X}{\cancel{a}} + \frac{\cancel{a}^2}{a}$$

$$P(X \geq a)$$

$$P\left(X + \frac{s^2}{a} \geq a + \frac{s^2}{a}\right)$$

$$\frac{s^2/\cancel{a}}{\frac{a^2+s^2}{\cancel{a}}}$$

$$P(y > \lambda) \leq$$

Q1) Basic inequality : $g \geq 0$ and $\uparrow$ on $[0,\infty)$ and g is an even function. X is any meas. funct^n then $\quad \mu(|X| \geq \lambda) \leq \dfrac{E(g(X))}{g(\lambda)} \quad ; \lambda > 0$

Proof :

$$E[g(X)] = \int g \circ X(\omega)\, d\mu(\omega)$$

$$= \int_{[|X|\geq\lambda]} g\circ X \cdot d\mu + \int_{[|X|<\lambda]} g\circ X \cdot d\mu$$

$$\geq \int_{[|X|\geq\lambda]} g\circ X\, d\mu \qquad \left\{ \begin{array}{l} \because\ g \geq 0 \Rightarrow \int g(\cdot) \geq 0 \\ \therefore \int_{[|X|<\lambda]} g(X(\omega))\, d\mu(\omega) \geq 0. \end{array} \right.$$

$$\geq \int_{[|X|\geq\lambda]} g(\lambda)\, d\mu \qquad \{\because g \text{ is even}\}$$

$$= g(\lambda)\ \mu([|X|\geq\lambda])$$

$$\therefore \quad E[g(X)] \geq g(\lambda)\, \mu([|X|\geq\lambda])$$

Use $\quad g(X) = |X|^r \qquad r > 1$

Here $g(X) \geq 0$ always $\quad g(-X) = |-X|^r = |X|^r = g(X) \equiv$ even funct^n

If $\quad X_1 \geq X_2 \Rightarrow |X_1| \geq |X_2| \quad$ since we only look at $[0,\infty)$

$\Rightarrow |X_1|^r \geq |X_2|^r$

$$\therefore \quad \mu(|X| \geq \lambda) \leq \frac{E(|X|^r)}{\lambda^r}$$

Let $r = 2$ & $X \equiv X - \mu$ then $\mu(|X-\mu| \geq \lambda) \leq \dfrac{E(|X-\mu|^2)}{\lambda^2} = \dfrac{V(X)}{\lambda^2}$

2] $\mu(\Omega) < \infty$ g is continuous & $X_n \xrightarrow{\mu} X$

Show $g(X_n) \xrightarrow{\mu} g(X)$

Proof g is continuous $\forall \varepsilon > 0\ \exists \delta > 0$ st $|x-y| < \delta \Rightarrow |g(x)-g(y)| < \varepsilon$

$\therefore \quad |X_n - X| < \delta \Rightarrow |g(X_n) - g(X)| < \varepsilon$

$$\mu(|X_n - X| < \delta) \leq \mu(|g(X_n) - g(X)| < \varepsilon)$$

$\therefore \quad \mu(\Omega) - \mu(|X_n - X| < \delta) \geqslant \mu(\Omega) - \mu(|g(X_n) - g(X)| < \varepsilon)$

ie $\quad \mu(|X_n - X| > \delta) \geqslant \mu(|g(X_n) - g(X)| > \varepsilon) \geq 0 \quad \{\because \mu(\Omega) < \infty$

$$0 \leq \lim_{n\to\infty} \mu(|g(X_n) - g(X)| > \varepsilon) \leq \lim_{n\to\infty} \mu(|X_n - X| > \delta)$$

$$\leq 0$$

$\therefore \quad \mu(|g(X_n) - g(X)| > \varepsilon) \longrightarrow 0 \qquad \forall \varepsilon > 0$

3] a) S_n = # of heads in 'n' indep coin tosses $\sim Bin(n, 1/2)$

$\frac{S_n}{n} \sim Ber(1/2)$

$E(S_n/n) = 1/2$ & $V(S_n/n) = 1/4n$

$P(|X - \mu| \geqslant 0.1) \leq \frac{1}{4n(0.1)^2}$ by Chebyshev's ineq

$P(|X - \mu| < 0.1) \geqslant 1 - \left(\frac{1}{0.04n}\right) \qquad \{\because P(\Omega) = 1 < \infty$

$$\lim_{n\to\infty} P(|X - \mu| < 0.1) \geqslant 1 - \lim_{n\to\infty} \left(\frac{1}{0.04n}\right) = 1$$

4) Jensens inequality : If g is convex on (a,b) st $\mu((a,b)) = \mu(\Omega) = 1$ and if $a < E(X) < b$ then $$g[E(X)] \leq E[g(X)]$$

Now $g(y) = |y|^p$ is convex when $p \geq 1$

Let $X = |y|$

$$|E(|y|)|^p \leq E(|y|^p) \implies [E(|y|)]^p \leq E(|y|^p) \quad \text{——} (*)$$

$\left\{ \because \begin{array}{l} |y| \geq 0 \\ E(|y|) \geq 0 \end{array} \right.$

To show $\|X\|_r$ is $\uparrow$ in r

$$\iff [E(|X|^r)]^{1/r} \geq [E(|X|^s)]^{1/s} \quad ; \; r > s > 0.$$

$$\iff E(|X|^r) \geq [E(|X|^s)]^{r/s}$$

$$\iff [E(|X|^{s\, r/s})] \geq [E(|X|^s)]^{r/s} \quad \text{——} (**)$$

Let $p \equiv r/s$ $\quad |y| \equiv |X|^s$ in $(*)$

we get $(**)$ and we are done.

#5] (a) Show $E(|X|^n) = 1/n \implies X$ is bdd a.s.

To find a bdd λ st $P(|X| > \lambda) = 0$

$$P(|X| \geq \lambda) \leq \frac{E(g(x))}{g(\lambda)}$$ { basic ineq, g is ≥ 0 $\uparrow$ on $[0,\infty)$ and even

$$\therefore \; P(|X| \geq \lambda) \leq \frac{E(|X|^n)}{\lambda^n}$$ Markov's inequality

$$0 \leq P(|X| \geq 1) \leq \frac{1/n}{1} \longrightarrow 0 \quad \text{as } n \to \infty$$

$$\therefore \; P(|X| \geq 1) \longrightarrow 0$$

$$\therefore \; P(|X| < 1) \longrightarrow 1$$

ie X is bdd a.e.

#6] $$P(X \geq a) = P\left(X + \frac{s^2}{a} \geq a + \frac{s^2}{a}\right)$$

$$\leq P\left(\left|X + \frac{s^2}{a}\right| \geq a + \frac{s^2}{a}\right)$$

$$\leq \frac{E\left(|X + s^2/a|^2\right)}{\left(a + \frac{s^2}{a}\right)^2} = \frac{E\left[\left(X + s^2/a\right)^2\right]}{\left[a\left(1 + s^2/a^2\right)\right]^2}$$

$$= \frac{E(X^2) + (s^2/a)^2}{a^2\left(1 + s^2/a^2\right)^2}$$

$$= \frac{\left(s^2 + s^4/a^2\right)}{a^2\left(1 + s^2/a^2\right)^2} = \frac{s^2\left(1 + s^2/a^2\right)}{a^2\left(1 + s^2/a^2\right)^2}$$

$$= \frac{s^2/a^2}{1 + s^2/a^2} = \frac{s^2}{a^2 + s^2}$$

7] $g_n : (\Omega, \mathcal{A}, \mu) \longrightarrow (\mathbb{R}, \mathcal{B})$ where $\Omega = [0,1]$, $\mathcal{A} \equiv \mathcal{B}_{[0,1]}$, $\mu = \lambda$

$$\lim_{n\to\infty} \int g_n(\omega)\, d\mu(\omega) = ? \qquad \text{where } g_n(x) = \frac{1+nx^2}{(1+x^2)^n}$$

Sol^n ;

$$g_n(x) = \frac{1+nx^2}{(1+x^2)^n}$$

$$\lim_{n\to\infty} g_n(x) \underset{\substack{\uparrow \\ \text{l'hospital rule}}}{=} \lim_{n\to\infty} \frac{2nx}{n(1+x^2)^{n-1}\, 2x} = \lim_{n\to\infty} \left[\frac{1}{(1+x^2)^{n-1}}\right] = 0$$

Also
$$g_n(x) = \frac{1+nx^2}{1+nx^2+\binom{n}{2}(x^2)^2+\binom{n}{3}(x^2)^3+\cdots(x^2)^n} \leqslant 1 \text{ always}$$

$$\therefore \quad |g_n(x)| \leqslant 1$$

Also $1 \in \mathcal{L}_1$ since $\int_\Omega 1\, d\mu(\omega) = \mu([0,1]) = 1 < \infty$

$\therefore$ since $g_n(x) \longrightarrow 0$ (which mean $\xrightarrow{a.e}$ also) by the DCT

we have $\displaystyle\int_0^1 g_n(\omega)\, d\mu(\omega) \longrightarrow 0$

Now $\displaystyle\int_0^1 g_n(x)\, d\mu = \int_0^1 g_n(x)\, d\lambda = \int_0^1 g_n(x)\, dx$

If g is a continuous funct^n on $[a,b]$
$\displaystyle\int_a^b g_n(x)\, dx = \int_a^b g_n(x)\, d\lambda$
R-S int $\equiv$ L-S integ

$$\therefore \lim_{n\to\infty} \int_0^1 g_n(x)\, dx = \lim_{n\to\infty} \int_0^1 g_n(x)\, d\mu$$

$$= 0$$

8] $X_n \xrightarrow{\mu} a$ and $Z_n \xrightarrow{\mu} b$

$$\Rightarrow \quad \sqrt{X_n} \xrightarrow{\mu} \sqrt{a}$$

$$Z_n^{\,3} \xrightarrow{\mu} b^3$$

$\left\{ \begin{array}{l} \text{for } \mu(\Omega) < \infty \text{ and } g \text{ continuous} \\ X_n \xrightarrow{\mu} X \Rightarrow g(X_n) \longrightarrow g(X) \end{array} \right.$

Also $Y_n \xrightarrow{d} Z$

$\therefore$ by Slutsky's theorem $\quad Y_n \sqrt{X_n} + Z_n \xrightarrow{d} \sqrt{a}\, Z + b^3$