Midterm 1 – STA 5446

Fall 2005 ; Oct. 6th.

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- Please write your name here.

  MAHTAB MARKER

- Please provide <u>detailed</u> solutions to all problems.

- Maximum number of points = 40.

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1. Definitions

[1] a) Define $\mathcal{A}$, if $\mathcal{A}$ is a σ-algebra of sets from Ω (and Ω is a generic set).

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[1] b) Let Ω be a generic space and $\mathcal{A}$ be a σ-field on Ω. Let $\mu : \mathcal{A} \to [0, \infty)$ having $\mu(\emptyset)$. Under which additional condition is μ called a measure?

[2] c) If $\{A_n\}_{n \geq 1}$ is a sequence of sets, define $\underline{\lim}\, A_n$ and $\overline{\lim}\, A_n$.

[2] d) Define $\mathcal{B}$, the σ-field of Borel sets.

[1] e) If $\{X_n\}_n$ is a sequence of measurable functions from $(\Omega, \mathcal{A}, \mu)$ to $(\mathbb{R}, \mathcal{B})$, define $X_n \xrightarrow[n \to \infty]{a.e.} X$.

Q1 (a) $\mathcal{A}$ is a collection of subsets A of a non-void set Ω is called a σ-field if

(i) $\Omega \in \mathcal{A}$

(ii) $A \in \mathcal{A} \rightarrow A^c \in \mathcal{A}$ (closed under complements)

(iii) $A_1, A_2 \ldots \in \mathcal{A} \rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$ (closed under countable union

(b) $\mu : (\Omega, \mathcal{A}) \rightarrow [0,\infty)$ is called a measure (countably additive) if

(i) $\mu(\phi) = 0$

(ii) $\mu(A) \geq 0$ for any $A \in \mathcal{A}$

(iii) $\mu\left(\sum_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$ for any disjoint union $\sum_{i=1}^{\infty} A_i \in \mathcal{A}$

(c) $\{A_n\}_{n \geq 1}$ is a sequence of sets

$$\underline{\lim}\, A_n = \bigcup_{n=1}^{\infty} \bigcap_{m \geq n}^{\infty} A_m$$

$$\overline{\lim}\, A_n = \bigcap_{n=1}^{\infty} \bigcup_{m \geq n}^{\infty} A_m$$

(d) Let $\mathcal{C}_I = \{ (a,b], (-\infty, b], (a,\infty) : a,b \in \mathbb{R} \}$

$\mathcal{C}_F = \{$ countable disjoint unions of elements of $\mathcal{C}_I \}$

$\mathcal{B} = \sigma[\mathcal{C}_F]$ is the borel set [on $\mathbb{R}$]

$\mathcal{B}$ is also characterized as $\sigma[\{(\text{open set})\}]$ or $\sigma[\{(\text{closed sets})\}]$

(e) $\{X_n\}$ is a sequence of measurable functions from $(\Omega, \mathcal{A}, \mu) \rightarrow (\mathbb{R}, \mathcal{B})$

$$X_n \xrightarrow[n \to \infty]{a.e} X \iff X_n(\omega) = X(\omega) \quad \forall \ \omega \notin N \text{ and } \mu(N) = 0$$

2. Results (No proofs required).

[2] a). State the monotone property of measures.

[2] b) State the "Correspondence theorem". (The theorem that establishes a 1 to 1 correspondence between L-S measures and gdfs.).

[3] c) State at least 2 results that are equivalent to $X_n \xrightarrow[n \to \infty]{a.e.} X$.

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2 (a) Monotone Property of Measures states

$\mu : (\Omega, \mathcal{A}) \to (\mathbb{R}, \mathcal{B})$ and $A_1, A_2 \cdots \in \mathcal{A}$

(i) $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n)$ for any $\{A_n\}_{n\geq 1} \uparrow$ seq

(ii) If $\mu(\Omega) < \infty$ then $\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n)$ for any $\{A_n\}_{n\geq 1} \downarrow$ seq

(iii) $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n)$ for any $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$ ✓

2 (b) Correspondence theorem :

$$\mu((a,b]) = F((a,b]) = F(b) - F(a)$$

establishes a 1-1 correspondence between the L-S measure μ on $\mathbb{R}$ and the generalized distⁿ function F on $\mathbb{R}$.

μ is the L-S measure on $\mathbb{R}$ which assigns finite values to finite intervals

F is the gdf on $\mathbb{R}$ which is a finite valued, increasing, right continuous function. ✓

2 (c) (i) $X_n \xrightarrow[n\to\infty]{a.e} X \iff X_m - X_n \xrightarrow[m \wedge n \to \infty]{a.e} 0$

(ii) $X_n \xrightarrow[n\to\infty]{a.e} X \iff \mu\left[\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} \left(|X_m - X_n| > \varepsilon\right)\right] = 0 \quad \forall \varepsilon > 0$

(iii) If $\mu(\Omega) < \infty$ then $X_n \xrightarrow[n\to\infty]{a.e} X \iff \mu\left[\max_{n \leq m \leq N} \left(|X_m - X_n| > \varepsilon\right)\right] \leq \varepsilon \quad \forall \varepsilon > 0$

and $N \geq n \geq n_\varepsilon$

[5] 3/ Let $X : \Omega \to [0, n]$ be a measurable function.

Define the sequence of functions

$$X_n(\omega) = \sum_{K=1}^{n \cdot 2^n} \frac{K-1}{2^n} \mathbb{1}_{\left[\left\{\omega \in \Omega \,\middle|\, \frac{K-1}{2^n} \leq X(\omega) < \frac{K}{2^n}\right\}\right]}(\omega).$$

Show that

$$X_n(\omega) \xrightarrow[n \to \infty]{} X(\omega), \text{ for each } \omega \in \Omega.$$

3] $X : \Omega \to [0, n)$ is a measurable function.

Define $X_n(\omega) = \sum_{k=1}^{n2^n} \frac{k-1}{2^n} I_{\left[\frac{k-1}{2^n} \leq X(\omega) < \frac{k}{2^n}\right]}(\omega)$.

$\mathcal{C} = \left\{ \omega \in \Omega \;\middle|\; \frac{k-1}{2^n} \leq X(\omega) < \frac{k}{2^n} \right\}_{k=1}^{n2^n}$ partitions Ω

$X(\omega) = \sum_{k=1}^{n2^n} X(\omega) I_{\left[\frac{k-1}{2^n} \leq X(\omega) < \frac{k}{2^n}\right]}(\omega)$ .... For ω arbitrary fixed

$\therefore \quad X_n(\omega) - X(\omega) = \sum_{k=1}^{n2^n} \left[\left(\frac{k-1}{2^n}\right) - X(\omega)\right] I_{\left[\frac{k-1}{2^n} \leq X(\omega) < \frac{k}{2^n}\right]}(\omega)$

$$|X_n(\omega) - X(\omega)| = \left| \sum_{k=1}^{n2^n} \left[\left(\frac{k-1}{2^n}\right) - X(\omega)\right] I_{\left[\frac{k-1}{2^n} \leq X(\omega) < \frac{k}{2^n}\right]}(\omega) \right|$$

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$$\leq \sum_{k=1}^{n2^n} \left|\left(\frac{k-1}{2^n}\right) - X(\omega)\right| I_{\left[\frac{k-1}{2^n} \leq X(\omega) < \frac{k}{2^n}\right]}(\omega)$$

Now ω belongs to exactly one of the intervals of $\mathcal{C}$ and so only one term in the above sum is non zero. The max absolute value will be the length of the interval to which $X(\omega)$ belongs. All intervals have length $1/2^n$

$\therefore \quad |X_n(\omega) - X(\omega)| \leq \frac{1}{2^n} \longrightarrow 0$ as $n \to \infty$

$\therefore \quad X_n(\omega) \xrightarrow[n \to \infty]{} X(\omega) \quad \forall\, \omega \in \Omega$

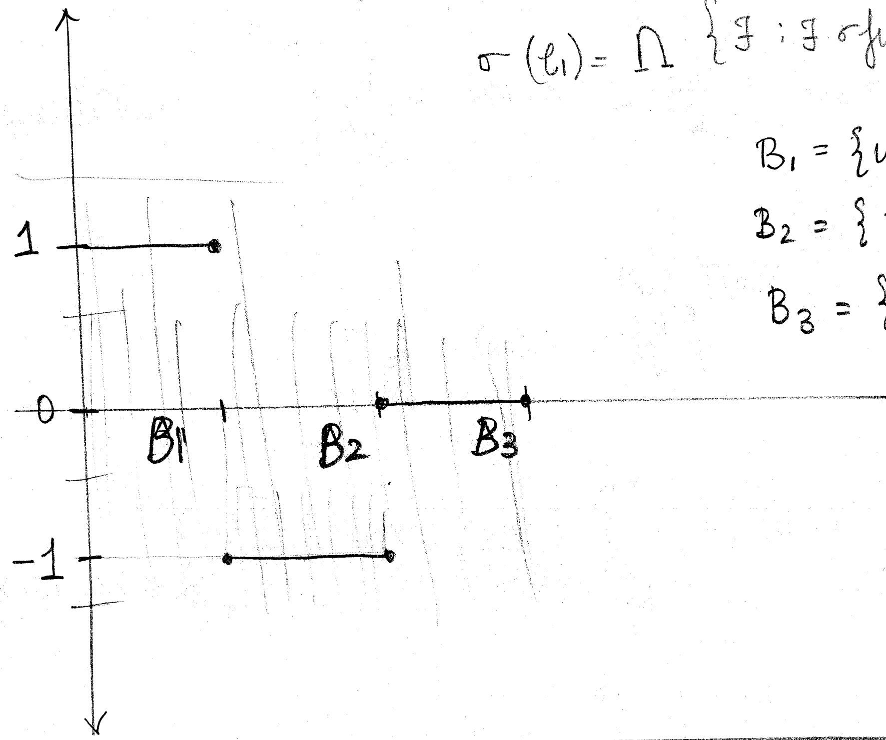
ROUGH

$\sigma(\{1,2\}) = \{\{1,2\}, \{3\}, \phi, \Omega\}$

$\Omega = \{1,2,3\}$

$\sigma((\{1\},\{2\})) = \{\{1\},\{2\},\{1,2\},\{1\}^c,\{1,2\}^c,\{2\}^c,\phi,\Omega\}$

$\sigma(\mathcal{C}_1) = \bigcap \{\mathcal{F} : \mathcal{F}\ \sigma\text{-field} \ \&\ \mathcal{C}_1 \subset \mathcal{F}\}$



$B_1 = \{\omega_1, \omega_6\}$

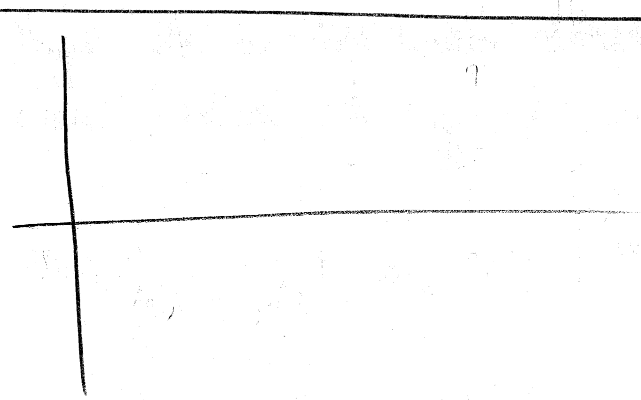
$B_2 = \{\omega_2, \omega_4\}$

$B_3 = \{\omega_3, \omega_5\}$

If $\Omega = \{1,2,3\}$

$\sigma(\{\{1\},\{2\}\}) = \{\{1\},\{2\},\{1\}^c,\{2\}^c,\{1,2\},\{1,2\}^c,\phi,\Omega\}$

$\therefore \sigma(\{\{A_1\},\{A_2\},\{A_3\}\}) = \{\{A_1\},\{A_2\},\{A_3\},\{A_1\}\cup\{A_2\},\{A_1\}\cup\{A_3\},$
$\{A_2\}\cup\{A_3\},\phi,\Omega\}$



[12] 4. Let $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6\}$.

- Let $\mathcal{K}_1 = 2^{\Omega}$ (all the subsets of Ω).
- Let $\mathcal{K}_2 = \sigma[\mathcal{C}_1]$, where

$\mathcal{C}_1 = \{A_1, A_2, A_3\}$ and

$A_1 = \{\omega_1\}$, $A_2 = \{\omega_2\} \cup \{\omega_4\} \cup \{\omega_6\}$,

$A_3 = \{\omega_3\} \cup \{\omega_5\}$.

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Define: $X : \Omega \to \mathbb{R}$

ω	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
$X(\omega)$	1	-1	0	-1	0	1

[6] a) Find $X^{-1}((-\infty, x))$ for all $x \in \mathbb{R}$.

That is, find $\{\omega \in \Omega \mid X(\omega) < x\}$.
Note that this set will change with x.
For example, $X^{-1}((-\infty, x)) = \emptyset$, if $x \leq -1$.

[3] b) Is X $\mathcal{B}$-$\mathcal{K}_1$ measurable? Yes!

[3] ~~c)~~ Is X $\mathcal{B}$-$\mathcal{K}_2$ measurable? No!

\# State the measurability criterion that you are using.

4] $\mathcal{K}_1 = 2^{\Omega}$ (8)

$$X(\omega) = (1)\,I_{\{\omega_1,\omega_6\}}(\omega) + (-1)\,I_{\{\omega_2,\omega_4\}} + (0)\,I_{\{\omega_3,\omega_5\}}(\omega)$$

(a) $X^{-1}((-\infty, x)) = \{\omega \in \Omega \mid X(\omega) < x\}$

$$= \begin{cases} \phi & \text{if} \quad x \leq -1 \\ \{\omega_2, \omega_4\} & \text{if} \quad -1 < x \leq 0 \\ \{\omega_2, \omega_3, \omega_4, \omega_5\} & \text{if} \quad 0 < x \leq 1 \\ \Omega & \text{if} \quad 1 < x \end{cases}$$

(b) X will be $\mathcal{B}-\mathcal{K}_1$ measurable if $X^{-1}((-\infty, x)) \subset \mathcal{K}_1$

ϕ, $\{\omega_2, \omega_4\}$, $\{\omega_2, \omega_3, \omega_4, \omega_5\}$, Ω are all $\subset \mathcal{K}_1$

$\therefore$ X is $\mathcal{B}-\mathcal{K}_1$ measurable

(c) X is $\mathcal{B}-\mathcal{K}_2$ measurable if $X^{-1}((-\infty, x)) \subset \mathcal{K}_2$

$\phi \subset \mathcal{K}_2$

But $\{\omega_2, \omega_4\} \not\subset \mathcal{K}_2$

$\therefore$ X is not $\mathcal{B}-\mathcal{K}_2$ measurable

[8] 5. • Let Ω be an arbitrary set.
• Let $A \subset \Omega$ be a subset of Ω.
• Let $\mathcal{F}$ be a σ-field on Ω.

• Let $X : A \longrightarrow \Omega$,

$$X(\omega) = \omega \text{ for any } \omega \in A.$$

[3] 1) Let $B \in \mathcal{F}$ be an arbitrary set in $\mathcal{F}$.
Verify that $X^{-1}(B) = A \cap B, \quad \forall\, B \in \mathcal{F}$.

[5] 2) Define now $\tilde{\mathcal{F}} = \{ A \cap B \mid B \in \mathcal{F} \}$.
(Hence $\tilde{\mathcal{F}}$ is obtained by intersecting the given set A with <u>all</u> $B \in \mathcal{F}$, in turn.)

Use 1) above and Proposition 2.1.2 (Preservation of σ-fields) to argue that $\tilde{\mathcal{F}}$ is a σ field on A.

OR Prove directly that $\tilde{\mathcal{F}}$ defined in 2) is a σ-field on A. [8].

5] Ω is arbitrary $A \subset \Omega$ $\mathcal{F}$ is a σ-field on Ω

$X : A \to \Omega$ st $X(\omega) = \omega$ for any $\omega \in A$

(1) $B \in \mathcal{F}$ is an arbitrary set in $\mathcal{F}$

$X^{-1}(B) = \{\omega \in A \mid X(\omega) \in B\}$ where $B \in \mathcal{F}$

$= \{\omega \in A \mid \omega \in B\}$ $\because X(\omega) = \omega$

Let $D = A \cap B = \{\omega \mid \omega \in A \ \& \ \omega \in B\}$

If $\omega_0 \in X^{-1}(B) \Rightarrow \omega_0 \in A \ \& \ \omega_0 \in B \Rightarrow \omega_0 \in D \Rightarrow X^{-1}(B) \subset D$

If $\omega_0 \in D \Rightarrow \omega_0 \in A \ \& \ \omega_0 \in B \Rightarrow \omega_0 \in X^{-1}(B) \Rightarrow D \subset X^{-1}(B)$

$\therefore X^{-1}(B) = A \cap B$

(2) $\tilde{\mathcal{F}} = \{A \cap B \mid B \in \mathcal{F}\} = X^{-1}(B)$ where $B \in \mathcal{F}$

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(i) $A = A \cap \Omega$ where $\Omega \in \mathcal{F} \Rightarrow A \in \tilde{\mathcal{F}}$

(ii) $C \in \tilde{\mathcal{F}} \to C$ is of the form $C = A \cap B$ where $B \in \mathcal{F}$

$C^c = A \cap B^c$ and $B^c \in \mathcal{F}$ (since $\mathcal{F}$ is a σ-field)

$\therefore C^c \in \tilde{\mathcal{F}}$

(iii) $C_1, C_2, \dots \in \tilde{\mathcal{F}} \Rightarrow$ $\left.\begin{array}{l} C_1 = A \cap B_1 \\ C_2 = A \cap B_2 \\ \vdots \end{array}\right\}$ for some $B_1, B_2 \dots \in \mathcal{F}$

$$\bigcup_{n=1}^{\infty} C_n = \bigcup_{n=1}^{\infty} (A \cap B_n) = A \cap \left(\bigcup_{n=1}^{\infty} B_n\right)$$

$B_1, B_2 \dots \in \mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} B_n \in \mathcal{F}$ since $\mathcal{F}$ is a field

$\therefore \bigcup_{n=1}^{\infty} C_n \in \tilde{\mathcal{F}}$

$\therefore \tilde{\mathcal{F}}$ is a σ-field

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Midterm 2 - Probability and Measure

STA 5446 - Nov. 17 2005

Very nice work!

1. Please write your name here.

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2. Please provide detailed solutions to all questions.

- Total number of points = 38

13 pages in all

(I) Definitions

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1) If $\{X_n\}_n$ and X are measurable functions define $X_n \xrightarrow[n\to\infty]{\mu} X$. [2]

2) Let $(\Omega, \mathcal{A}, \mu)$ be a fixed measure space, and let $X : (\Omega, \mathcal{A}) \to (\mathbb{R}, \mathcal{B})$ denote a measurable function. [3]
Define $\int X d\mu$, the Lebesgue integral of X with respect to μ.

3) Define "convergence in distribution"; specify the context. [2]

4) Let $X : (\Omega, \mathcal{A}, \mu) \to (\Omega', \mathcal{A}')$ be a measurable function. Define μ', the induced measure of X. [1].

(I) Definitions

1) If $\{X_n\}$ is a seq of a.e finite measurable functions and X is a measurable function

$$X_n \xrightarrow[n\to\infty]{\mu} X \quad \text{if} \quad \mu\left([|X_n - X| \geq \varepsilon]\right) \xrightarrow[n\to\infty]{} 0 \qquad \forall\ \varepsilon > 0$$

2) $(\Omega, \mathcal{A}, \mu)$ is a measure space.

$X : (\Omega, \mathcal{A}) \longrightarrow (\mathbb{R}, \mathcal{B})$ is a $\mathcal{B}$-$\mathcal{A}$ measurable function.

i) If X is a simple funct^{n} and $X \geq 0$ ie $X = \sum_{i=1}^{n} x_i I_{A_i}$

where $x_i \geq 0$ and $x_i \in \mathbb{R}$, $A_i \in \mathcal{A}$ st $\sum_{i=1}^{n} A_i = \Omega$

then $\int X \cdot d\mu \equiv \sum_{i=1}^{n} x_i \, \mu(A_i)$

ii) If $X \geq 0$ meas function then

$$\int X \cdot d\mu \equiv \sup\left\{ \int Y \cdot d\mu \ : \ 0 \leq Y \leq X \text{ and } Y \text{ is simple funct}^{n} \right\}$$

iii) If X is any arbitrary measurable function then $X = X^{+} - X^{-}$

$$\int X \, d\mu \equiv \int X^{+} d\mu - \int X^{-} d\mu$$

3) If $\{X_n\}$ is a seq of random variables with $X_n \cong F_n$

X is a random variable with $X_0 \cong F_0$. We say

$X_n \xrightarrow[n\to\infty]{d} X_0$ if $\lim_{n\to\infty} F_n(x) = F_0(x)$ $\forall$ $x \in$ set of continuity pts of $F_0(\cdot)$

$\boxed{\text{PTO}}$

4). $X : (\Omega, \mathcal{A}, \mu) \longrightarrow (\Omega', \mathcal{A}')$ is a measurable functⁿ. The measure induced by X is : $\mu'(A') = \mu_X(A') \equiv \mu(X^{-1}(A')) \quad \forall\ A' \in \mathcal{A}'$

II Statements/Results

8/8 + 2 bonus points for the example!

1) Let $\{X_n\}_n$ and X be measurable functions defined on $(\mathcal{F}, \mathcal{K}, \nu)$. Assume $\nu(\mathcal{F}) < \infty$. a) What is the connection between $X_n \xrightarrow[n\to\infty]{a.e.} X$ and $X_n \xrightarrow[n\to\infty]{\nu} X$? [1].

b) Is the condition $\nu(\mathcal{F}) < \infty$ needed essentially for a) to hold? [1].

Give example

2) Assume again that $\{X_n\}_n$ are measurable functions, now on $(\mathcal{F}, \mathcal{K}, P)$, where P is a probability measure.

Assume that $X_n \xrightarrow[n\to\infty]{d} 10$.

What can you conclude about the convergence in probability of X_n? State the result you are using [2].

(II) Statements and Results

1) a) $\{X_n\}$ and X are measurable funct^n defined on $(\mathcal{F}, \mathcal{K}, \nu)$ where $\nu(\mathcal{F}) < \infty$

then $X_n \xrightarrow[n\to\infty]{a.e} X \Rightarrow X_n \xrightarrow[n\to\infty]{\nu} X$

ie if $\nu(\mathcal{F}) < \infty$ then convergence almost everywhere implies convergence in measure

b) <u>Yes</u>! the condition $\nu(\mathcal{F}) < \infty$ is needed for the result to hold

If $\nu(\mathcal{F}) = \infty$ then $X_n \xrightarrow[n\to\infty]{a.e} X$ need not imply $X_n \xrightarrow[n\to\infty]{\nu} X$

<u>Example</u> Let $\mathcal{F} = [0,\infty)$

$\mathcal{K} = \mathcal{B}_{[0,\infty)}$

$\nu = \lambda$ (Lebesgue measure)

Define $X_n = I_{[n, n+1)}$; $n = 0, 1, 2 \ldots$

Define $X(\omega) = 0 \quad \forall\ \omega \in \mathcal{F}$

Here $\nu(\mathcal{F}) = \lambda([0,\infty)) = \infty$

Let $\omega_0 \in \mathcal{F}$ be arbitrary fixed $\Rightarrow \omega_0 \in A_{n_0} = [n_0, n_0+1)$ for some n_0

$\therefore X_{n_0}(\omega_0) = 1$ and $X_n(\omega_0) = 0 \quad \forall\ n \neq n_0$

$\therefore \lim_{n\to\infty} X_n(\omega_0) = 0 = X(\omega_0)$

$\therefore X_n \xrightarrow[n\to\infty]{a.e} X$

Now $\quad \nu(|X_n - X| \geq \varepsilon) = \nu(|X_n| \geq \varepsilon) \quad \{\because X = 0$

$$= \nu\left(I_{[n,n+1)} \geq \varepsilon\right)$$

$$= \nu\left(I_{[n,n+1)} = 1\right)$$

$$= \nu([n, n+1))$$

$$= \lambda([n, n+1)) \quad \{\because \nu \equiv \lambda$$

$$= 1 \not\longrightarrow 0 \quad \text{as } n \to \infty$$

$\therefore$ Here $\nu(\mathcal{F}) = \infty$

$$X_n \xrightarrow[n\to\infty]{a.e} X \quad \text{but} \quad X_n \underset{n\to\infty}{\overset{\nu}{\not\longrightarrow}} X$$

2) $\{X_n\}$ is a seq of measurable functions defined on $(\mathcal{F}, \mathcal{K}, P)$
P is a probability meas ie $P(\mathcal{F}) = 1$ then we have

$$X_n \xrightarrow[n\to\infty]{d} a \iff X_n \xrightarrow[n\to\infty]{P} a \qquad \{a \equiv \text{constant}$$

$$\therefore \text{ if } X_n \xrightarrow[n\to\infty]{d} 10 \implies X_n \xrightarrow[n\to\infty]{P} 10$$

3) <u>Dominated Convergence Theorem</u>
$\{X_n\}$ is a seq of measurable funct^n st $|X_n| \leq Y$
for some $Y \in \mathcal{L}_1 \equiv \{g : \int |g| < \infty\}$ if one of the cond^n holds :

i) If $X_n \xrightarrow[n\to\infty]{a.e} X$ then $\int |X_n - X| \, d\mu \xrightarrow[n\to\infty]{} 0$

OR ii) if $X_n \xrightarrow[n\to\infty]{\mu} X$ then $\int |X_n - X| \, d\mu \xrightarrow[n\to\infty]{} 0$

PTO

(II) 4) Theorem of the Unconscious Statistician:

If $X: (\Omega, \mathcal{A}, \mu) \longrightarrow (\Omega', \mathcal{A}', \mu_X)$ is a $\mathcal{A}'$-$\mathcal{A}$ meas function

$g: (\Omega', \mathcal{A}') \longrightarrow (\mathbb{R}, \mathcal{B}, \mu_{g(x)})$ is a $\mathcal{B}$-$\mathcal{A}'$ meas funct^{n}

$\mu_X = \mu(X^{-1}(A'))$ for $A' \in \mathcal{A}'$ is the measure induced by X.

then

1) μ_X completely determines the measure $\mu_{g(x)}$ on $(\mathbb{R}, \mathcal{B})$

ie $\mu_{g(x)}(B) = \mu_X(g^{-1}(B))$ $\forall B \in \mathcal{B}$

2) $\displaystyle\int_{X^{-1}(A')} g \circ X \, d\mu = \int_{A'} g \, d\mu_X$ for $A' \in \mathcal{A}'$

~~3)~~ State the dominated convergence theorem. [2]

~~4)~~ State the theorem of the unconscious statistician. [2].

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III

Let $X \in \mathcal{L}_1$.

~~1)~~ Argue that $\int |X| \, 1_{[|X| \leq n]} \nearrow_{n \to \infty} \int |X|$. [2].
(State the result you are using.)

~~2)~~ Use 1) to show that [6].

$\forall \varepsilon > 0$ there exists an appropriate $\delta_\varepsilon > 0$ such that
<u>if</u> $\mu(A) < \delta_\varepsilon$ <u>then</u> $\int_A |X| d\mu < \varepsilon$.

(III) $X \in \mathcal{L}_1$

1) $$\int |X| \, d\mu = \int_{[|X| \leq n]} |X| \, d\mu + \int_{[|X| > n]} |X| \, d\mu$$

Now $\lim_{n \to \infty} [|X| \leqslant n] = \Omega$

Also: $[|X| \leqslant n] \equiv A_n$ is $\uparrow$ as $n \uparrow$

$\therefore$ $I_{[|X| \leqslant n]} \uparrow I_{\Omega}$ as $n \longrightarrow \infty$

$\therefore$ $|X| \, I_{[|X| \leqslant n]} \uparrow |X| \, I_{\Omega}$ as $n \to \infty$

Now $\left\{ |X| \, I_{[|X| \leqslant n]} \right\} \geqslant 0$ always

$\therefore$ by MCT we have. $\int |X| \, I_{[|X| \leqslant n]} \uparrow \int |X|$ as $n \to \infty$

PTO

2) From 1) we have $\lim_{n\to\infty} \int |X| \, I_{[|X| \leq n]} \, d\mu = \int |X| \, d\mu$

$$\int |X| \, d\mu = \int_{[|X| \leq n]} |X| \, d\mu + \int_{[|X| > n]} |X| \, d\mu$$

$$\lim_{n\to\infty} \int |X| \, d\mu = \lim_{n\to\infty} \int_{[|X| \leq n]} |X| \, d\mu + \lim_{n\to\infty} \int_{[|X| > n]} |X| \, d\mu$$

ie $\int |X| \, d\mu = \int |X| \, d\mu + \lim_{n\to\infty} \int_{[|X| > n]} |X| \, d\mu$ $\left\{ \begin{array}{l} \text{and we can} \\ \text{cancell} \because X \in \mathcal{L}_1 \end{array} \right.$

$\therefore \lim_{n\to\infty} \int_{[|X| > n]} |X| \, d\mu = 0 \Rightarrow \forall \varepsilon > 0 \ \exists \ n_\varepsilon$ st $\int_{[|X| > n_\varepsilon]} |X| \, d\mu < \frac{\varepsilon}{2}$

Now if $\forall \ \varepsilon > 0 \ \exists \ \delta_\varepsilon$ st $\mu(A) < \delta_\varepsilon$

$$\int_A |X| \, d\mu = \int_A |X| \, I_{[|X| \leq n_\varepsilon]} + \int_A |X| \, I_{[|X| > n_\varepsilon]}$$

$$\leq \int_A |X| \, I_{[|X| \leq n_\varepsilon]} \, d\mu + \int |X| \, I_{[|X| > n_\varepsilon]} \qquad \{ \because A \subset \Omega$$

$$\leq \int n_\varepsilon I_A \, d\mu + \int_{[|X| > n_\varepsilon]} |X| \, d\mu$$

$$< n_\varepsilon \, \mu(A) + \frac{\varepsilon}{2}$$

$$= n_\varepsilon \, \delta_\varepsilon + \frac{\varepsilon}{2}$$

$\therefore \int |X| \, d\mu < \varepsilon$ by taking $\delta_\varepsilon = \frac{\varepsilon}{2 n_\varepsilon}$

(IV) Let $X \geq 0$, $X \in \mathcal{L}_2(P)$.

Let $a > 0$ such that $EX > a$. [6]

Apply the Cauchy-Schwarz inequality to the variable $X \cdot 1_{[X>a]}$ and conclude that:

$$P(X>a) \geq \frac{(EX-a)^2}{EX^2}$$

(V) Let $\Omega = [0,1]$. Let λ be the Lebesgue measure on $[0,1]$.

Let $A_1 = [0, \frac{1}{2})$

$A_2 = [\frac{1}{2}, 1]$ [4]

$A_3 = [0, \frac{1}{3})$

$A_4 = [\frac{1}{3}, \frac{2}{3})$

$A_5 = [\frac{2}{3}, 1]$

$\vdots$

Define $X_n(\omega) = 1_{A_n}(\omega)$, for $n = 1, 2, \ldots$.

Define $X(\omega) = 0$, for all $\omega \in \Omega$.

Show that $\{X_n\}_n$ converges in measure to X.

$$E(X) - a = E\left(X I_{[X \geq a]}\right)$$

$$= \int_{[X>a]} X + \int_{[X<a]} X - \int_{[X<a]} X$$

$$E(X) - \int X$$

(IV) $X \geq 0$, $X \in \mathcal{L}_2(P)$

Let $a > 0$ st $E(X) > a$

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Cauchy Schwarz inequality states

$$E(|MN|) \leq [E(|N|^2)]^{1/2}[E(|M|^2)]^{1/2} \qquad (*)$$

Let $M \equiv X$

and $N \equiv I_{[X>a]}$ in the Cauchy-Schwartz ineq

$$E(|MN|) = E\left(X I_{[X>a]}\right) = E\left(X I_{[X-a>0]}\right) = \int_{[X>a]} X \, dP$$

$$E(|MN|) = \int_{[X>a]} X \, dP - \int_{[X<a]} X \, dP + \int_{[X<a]} X \, dP = E(X) - \int_{[X<a]} X \, dP$$

$$\therefore E(|MN|) \geq E(X) - \int a \, dP = E(X) - a$$

now $E^{1/2}(|M|^2) = \left[E\left(I_{[X>a]}\right)^2\right]^{1/2}$

$$E\left(I_{[X>a]}\right) = \int I_{[X>a]} \, dP = \int_{[X>a]} dP = P(X>a)$$

Now $E^{1/2}(|N|^2) = \left[E(X^2)\right]^{1/2}$

$\therefore$ we have $(E|MN|)^2 \overset{\geq}{\cancel{>}} (E(X)-a)^2$ ← but the result will still hold

$$E(|N|^2) = E(X^2)$$

$$E(|M|^2) = P(X>a)$$

$$\therefore (E(X)-a)^2 \leq E(X^2) \cdot P(X>a) \qquad \{\text{from } (*) \text{ \& squaring}$$

$$\therefore P(X>a) \geq \frac{(E(X)-a)^2}{E(X^2)}$$

— Q.E.D

(V) $\Omega = [0,1]$ $\lambda =$ lebesgue measure on $[0,1]$

$A_1 = [0, 1/2)$

$A_2 = [1/2, 1]$

$A_3 = [0, 1/3)$

$A_4 = [1/3, 2/3)$

$A_5 = [2/3, 1]$

$\vdots$

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$X_n(\omega) = I_{A_n}(\omega) \quad n = 1, 2, \cdots$

$X(\omega) = 0 \quad \forall\, \omega \in \Omega$

To show $X_n \xrightarrow{\mu} X$ ie to show $\mu(|X_n - X| > \varepsilon) \to 0 \quad \forall \varepsilon > 0$

Consider $\mu(|X_n - X| > \varepsilon) = \mu(|X_n| > \varepsilon)$ $\{\because X = 0$

$= \mu([I_{A_n} > \varepsilon])$

$= \mu([I_{A_n} = 1])$ $\left\{ \begin{array}{l} \text{since } \varepsilon > 0 \\ \& \ I_{A_n} = 0 \text{ or } 1 \end{array} \right.$

$= \mu(A_n)$ $\left\{ \because I_{A_n} = \begin{array}{ll} 1 & \forall\, \omega \in A_n \\ 0 & \text{ow} \end{array} \right.$

Now since $\lambda =$ lebesgue measure

$\mu(A_n) =$ length of the interval A_n

$\lim_{n\to\infty} \lambda(A_n) \longrightarrow 0$ $\left\{ \begin{array}{l} \text{by the way we constructed} \\ \text{the } A_n\text{'s} \end{array} \right.$

$\therefore \ 0 \le \lim_{n\to\infty} \mu(|X_n - X| > \varepsilon) = 0$

ie $X_n \xrightarrow{\mu} X$

(VI) Let $\{S_n\}_n$ be a sequence of random variables such that: $ES_n = a$, for some $a \in \mathbb{R}$.

$\operatorname{var} S_n = \frac{c}{n}$, for some $c > 0$.

1) Use your favorite inequality to show that [2]

$$S_n \xrightarrow[n\to\infty]{P} a$$

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2) Let now $\{Y_n\}_n$ be a sequence of random variables such that $Y_n \xrightarrow{d} Y$.

Derive the limit in distribution of $\{S_n^2 \cdot Y_n\}_n$. [2].

(VI) $\{S_n\}_n$ is a seq of random variables st $E(S_n) = a$, $a \in \mathbb{R}$
$V(S_n) = \frac{c}{n}$, $c > 0$

1) To show $S_n \xrightarrow[n\to\infty]{P} a$ ie $P(|S_n - a| \geq \varepsilon) \xrightarrow[n\to\infty]{} 0 \quad \forall\, \varepsilon > 0$

Now $P(|X-\mu| \geq \lambda) \leq \frac{Var(X)}{\lambda^2} \quad \forall\, \lambda > 0$ [Chebyshev's ineq

$\therefore \quad P(|S_n - a| \geq \varepsilon) \leq \frac{c/n}{\varepsilon^2}$ for any $\varepsilon > 0$

ie $0 \leq P(|S_n - a| \geq \varepsilon) \leq \left(\frac{c}{\varepsilon^2}\right)\frac{1}{n}$ $\quad c > 0,\ \varepsilon > 0$

$$0 \leq \lim_{n\to\infty} P(|S_n - a| \geq \varepsilon) \leq \lim_{n\to\infty} \frac{c}{\varepsilon^2}\left(\frac{1}{n}\right)$$

$$0 \leq \lim_{n\to\infty} P(|S_n - a| \geq \varepsilon) \leq 0$$

ie $P(|S_n - a| \geq \varepsilon) \longrightarrow 0$ as $n \to \infty$ $\quad \forall\, \varepsilon > 0$

2) $\{Y_n\}$ is a seq of random variables st $Y_n \xrightarrow{d} Y$

Also $S_n \xrightarrow{P} a$ and $P(\Omega) < \infty \Rightarrow g(S_n) \xrightarrow{P} g(a)$

for any continuous function 'g'

$\therefore \quad S_n^2 \xrightarrow{P} a^2$

By Slutsky's theorem : given $Y_n \xrightarrow{d} Y$, $S_n^2 \xrightarrow{P} a^2 \Rightarrow S_n^2 Y_n \xrightarrow{d} a^2 Y$