CONSEQUENCES OF CONVERGENCE IN DIST$^{\underline{N}}$

Theorem: (Helly - Bray) : Let $(\Omega, \mathcal{A}, P)$ be a probability space
Let $X_n \xrightarrow{d} X$ (ie $F_n \longrightarrow F$)

1) If g is bdd and continuous a.s F (w.r.t a meas $\mu \equiv F$) then

$$E[g(X_n)] = \int g \, dF_n \xrightarrow[n \to \infty]{} E[g(X)] = \int g \, dF$$

[However note $X_n \xrightarrow{d} X \not\Rightarrow E(X_n) \longrightarrow E(X)$ always]

2) If $E[g(X_n)] \xrightarrow[n \to \infty]{} E[g(X)]$ for any bdd and continuous g'

then $X_n \xrightarrow{d} X$ ie $F_n \xrightarrow[n \to \infty]{} F$

Theorem: (Skorokhod) If $X_n \xrightarrow{d} X$ then $\exists$ $\{Y_n\}_n$ and Y defined on some $(\Omega, \mathcal{A}, P)$ such that

a) $X_n \cong Y_n$ (equal in dist$^{\underline{n}}$)

b) $X \cong Y$

c) $Y_n \xrightarrow[P]{a.s} Y$ (a.s. in prob)

{ eg: $X_1, X_2 \cdots X_n \sim \text{iid } f(x)$ }

Proof of Helly-Bray: Since $X_n \xrightarrow{d} X$ we use &

$E[g(X_n)] \equiv \int g \, d\mu_n = \int g \, dF_n$

Also since $X_n \cong Y_n$ we have $E[g(X_n)] = E[g(Y_n)]$

Since $X \cong Y$ we have $E[g(X)] = E[g(Y)]$

We need to show $\int g(Y_n) \, dP \longrightarrow \int g(Y) \, dP$

We use DCT and so we need to show ① $|g(Y_n)| \leq M$

② $\int |M| \, dP < \infty$

We have g is bdd & continuous a.s F $\Rightarrow |g(Y_n)| \leq M$ for some M

Also $\int |M| \, dP = |M| \int dP = |M| < \infty$

We have g is continuous a.s P_X

We also need to show $g(Y_n) \xrightarrow{a.s} g(Y)$

ie we need to show $g(Y_n(\omega)) \longrightarrow g(Y(\omega)) \;\; \forall \omega \in A$ st $P(A) = 1$

[$Y_n(\omega) = y_n$ some seq of # & we have $y_n \to y$ we need to show $g(y_n) \to g(y)$]

Let $A_1 \equiv \{\omega \in \Omega : Y_n(\omega) \longrightarrow Y(\omega)\}$

$\therefore P(A_1) = 1$ since $Y_n \xrightarrow{a.s} Y$

Let $A_2 \equiv \{\omega \in \Omega \mid g \text{ is cont at } Y(\omega)\}$

$$P(A_2) = P_Y(\{y \in R : g \text{ is continuous at } y\})$$
$$= P_X(\{y \in R : g \text{ is cont at } y\})$$
$$= 1.$$

$(\Omega, \mathcal{A}, P) \xrightarrow{Y} (\mathbb{R}, \mathcal{B}$

P_Y is the induce meas & has 1-1 corresp with F_Y

By construct^n $F_Y = F_X$

Take $A = A_1 \cap A_2$

$$0 \leq P(A^c) = P(A_1^c \cup A_2^c) \leq P(A_1^c) + P(A_2^c)$$

$$0 \leq P(A^c) \leq 0$$

$\rightarrow P(A^c) = 0$ & so $P(A) = 1$

by continuity $Y_n \xrightarrow{a.s} Y \Rightarrow g(Y_n) \xrightarrow{a.s} g(Y)$

Mann Wald theorem: If $X_n \xrightarrow{d} X$ and g is continuous a.s F then $g(X_n) \xrightarrow{d} g(X)$

eg: $X_n \xrightarrow{d} N(0,1)$ then $X_n^2 \xrightarrow{d} [N(0,1)]^2 = \chi_1^2$

Proof: $X_n \xrightarrow{d} X$ so $\exists \{Y_n\}$ & Y st

$X_n \cong Y_n$

$X \cong Y$

$Y_n \xrightarrow{a.s} Y$

$\therefore g(Y_n) \xrightarrow{a.s} g(Y)$ $\quad$ {by cont of g

$\Rightarrow g(Y_n) \xrightarrow{d} g(Y)$ $\quad$ $\{\xrightarrow{a.s} \Rightarrow \xrightarrow{d}\}$

$\Rightarrow g(X_n) \xrightarrow{d} g(X)$

DEF : $\boxed{\text{Convergence in } \mathcal{L}_r}$

X_n converges in r^{th} mean to X if $X_n \xrightarrow{\mathcal{L}_1} X$ ie $E(|X_n - X|^r) \longrightarrow 0$

Eg: for $r = 2 \Rightarrow E(X_n^2) \xrightarrow[n\to\infty]{} E(X^2)$

for $r = 1 \Rightarrow E(X_n) \longrightarrow E(X)$

UNIFORM INTEGRATION

Def1: $\{X_t\}_{t\in T}$ is called integrable if $\sup\limits_{t\in T} \{E|X_t|\} < \infty$

NOTE : It is possible to have $E|X_t| < \infty$ but $\sup\limits_{t\in T} \left(E|X_t|\right) = \infty$

eg if $E(|X_t|) = t$

Def 2 : $\{X_t\}_{t\in T}$ is called uniformly integrable (u.i) if

$$\sup_{t\in T} E|X_t| I_{[|X_t| \geq \lambda]} \xrightarrow[\lambda\to\infty]{} 0 \quad \left\{ \begin{array}{l} \text{ie } \forall \varepsilon > 0 \ \exists \lambda_\varepsilon \text{ st} \\ \sup\limits_{t\in T} E|X_t| I_{[|X_t| \geq \lambda]} \leq \varepsilon \quad \forall \lambda \geq \lambda_\varepsilon \end{array} \right.$$

ie if for large values of X_t the $E|X_t|$ is also large then the function is not integrable.

$\boxed{\text{Remark}}$: Suppose that $\{X_t\}_{t\in T}$ is such that $|X_t| \leq Y$, $\forall t\in T$ and $Y \in \mathcal{L}_1$ for some Y then $\{X_t\}$ is uniformly integrable

Proof

$$E|X_t| I_{[|X_t| > \lambda]} = \int_{[|X_t| > \lambda]} |X_t| \, d\mu \leq \int_{[|X_t| > \lambda]} |Y| \, d\mu$$

$$E\,|X_t|\,I_{[|X_t|>\lambda]} \leq \int_{[|y|>\lambda]} |y|\,d\mu$$

We now need to show $\mu[|y|>\lambda] \longrightarrow 0$ as $\lambda \to \infty$

Now $\mu[|y|>\lambda] \leq \frac{E\,|y|}{\lambda}$ by Markov ineq

$$\mu[|y|>\lambda] \leq \left(\frac{E\,|y|}{\lambda}\right) \longrightarrow 0 \quad \text{as } \lambda \to \infty$$

$$\therefore \sup_t E\,|X_t|\,I_{[|X_t|>\lambda]} \longrightarrow 0 \quad \text{as } \lambda \to \infty.$$

$\boxed{\text{THEOREM}}$ Uniform Absolute Continuity of the Integral

Assume $\underline{\mu(\Omega)<\infty}$ **

$\{X_t\}_{t\in T}$ is $\underline{u.i}$ $\Longleftrightarrow$ (a) $\{X_t\}_{t\in T}$ is integrable

(b) $\underline{\mu(A)<\delta_\varepsilon} \Rightarrow \underbrace{\sup_t \int_A |X_t|\cdot d\mu}_{= \sup_{t\in T} E\,|X_t| I_A} < \varepsilon$

Proof : "$\Longrightarrow$" a) $E\,|X_t| = E\,|X_t|\,I_{[|X_t|\leq\lambda]} + E\,|X_t|\,I_{[|X_t|>\lambda]}$

$$\leq \lambda\mu(\Omega) + E\,|X_t|\,I_{[|X_t|>\lambda]}$$

$$\leq \lambda\mu(\Omega) + \varepsilon$$

$$< \infty \qquad \{\text{since } \mu(\Omega)<\infty$$

"$\Longrightarrow$" b) Take A st $\mu(A) < \delta$ for δ to be chosen later.

Note $[|X_t| \geq \lambda]$ is one possible choice for A.

Consider $\displaystyle\int_A |X_t| \, d\mu = \int_A |X_t| I_{[|X_t| \geq \lambda]} + \int_A |X_t| I_{[|X_t| \leq \lambda]} \cdot d\mu.$

$$\leq \int_A |X_t| I_{[|X_t| \geq \lambda]} + \lambda \mu(A)$$

$$\leq \int_\Omega |X_t| I_{[|X_t| \geq \lambda]} + \lambda \mu(A).$$

$$\sup_{t \in T} \int_A |X_t| \, d\mu \leq \sup_{t \in T} \int |X_t| I_{[|X_t| \geq \lambda]} \, d\mu + \lambda \mu(A)$$

$$< \frac{\varepsilon}{2} + \lambda \mu(A) \qquad \{\text{by def of u.i}$$

$$< \frac{\varepsilon}{2} + \lambda \frac{\varepsilon}{2\lambda} \qquad \{\text{letting } \delta = \frac{\varepsilon}{2\lambda}$$

$$\therefore \sup_{t \in T} \int_A |X_t| \cdot d\mu < \varepsilon$$

"$\Longleftarrow$" Take $A = [|X_t| \geq \lambda]$

Consider $\mu(A) = \mu(|X_t| \geq \lambda) \leq \dfrac{E(|X_t|)}{\lambda}$

Now by (a) $\{X_t\}$ is integrable and so $E|X_t| < \infty \Rightarrow \dfrac{E|X_t|}{\lambda} \xrightarrow[\lambda \to \infty]{} 0$

$\therefore \mu(A) \longrightarrow 0$ and so by (b) we get $\displaystyle\sup_{t \in T} \int_A |X_t| \, d\mu < \varepsilon$

Theorem : (Vitali) Let $r > 0$, assume $\{X_n\}$ st $X_n \in \mathcal{L}_r \equiv \{g : \int |g|^r < \infty$

Assume $X_n \xrightarrow{\mu} X^*$ for some X and $\mu(\Omega) \overset{*}{<} \infty$ then the foll statements are equivalent

1) $\{|X_n|^r, n \geq 1\}$ are u.i

2) $X_n \xrightarrow{\mathcal{L}_r} X$

3) $E|X_n|^r \longrightarrow E|X|^r$

4) $\overline{\lim} \; E|X_n|^r \leq E|X|^r < \infty$

Corollary ($\mathcal{L}_1$ convergence)

$X_n \xrightarrow{\mathcal{L}_1} X \iff$ a) $X_n \xrightarrow{\mu} X$; $\mu(\Omega) < \infty$

b) $\{|X_n|, n \geq 1\}$ are u.i

Proof "$\Leftarrow$" $X_n \xrightarrow{\mu} X$ and $\{|X_n|\}$ are u.i

so by Vitali's theorem with $r = 1$ we get $X_n \xrightarrow{\mathcal{L}_1} X$

"$\Rightarrow$" a) $\mu(|X_n - X| > \varepsilon) \leq \dfrac{E|X_n - X|}{\varepsilon} \longrightarrow 0$

since $X_n \xrightarrow{\mathcal{L}_1} X$

$\Rightarrow X_n \xrightarrow{\mu} X$

$\therefore$ by Vitali's theorem $\Rightarrow \{|X_n|\}$ are u.i

Proof of Vitali's Theorem :

Plan: Show 1) $\Rightarrow$ 2) $\Rightarrow$ 3) $\Rightarrow$ 4) $\Rightarrow$ 1)

2) $\Rightarrow$ 3) trivial and done before.

3) $\Rightarrow$ 4) easy.

4) $\Rightarrow$ 1) Hard (read but no proof for exam)

Prove 1) $\Rightarrow$ 2)

To show X also $\in \mathcal{L}_r$

We have $X_n \xrightarrow{r} X \Rightarrow \exists\, n'$ st $X_{n'} \xrightarrow{a.e} X$ (by theorem

$$\Rightarrow |X_{n'}|^r \xrightarrow{a.s} |X|^r$$

Now $E(|X|^r) = E(\underline{\lim}\ |X^r|)$ (excercise)

$\leq \underline{\lim}\ E|X^r|$ (Fatou's lemma)

$< \infty$ ($\because \{|X|^r\}$ are u.i)

Now if $\{|X_n|^r\}$ is u.i and $X \in \mathcal{L}_r$ then $\{|X_n - X|^r\}$ is also u.i (excercise

To show now $E(|X_n - X|^r) \to 0$

$$E|X_n - X|^r = E|X_n - X|^r I_{[|X_n - X| \geq \varepsilon]} + E|X_n - X|^r I_{[|X_n - X| \leq \varepsilon]}$$

$$\leq E|X_n - X|^r I_{[|X_n - X| \geq \varepsilon]} + \varepsilon^r$$

$$< \varepsilon + \varepsilon^r$$ (excercise.

SHUVA

To show uniform integrability we need to show

$$\sup E|X_n| I_{[|X_n|>\lambda]} \longrightarrow 0 \quad \text{as } \lambda \to \infty$$

$$\Updownarrow$$

ⓐ $\sup_n E|X_n| < \infty$

ⓑ $\forall \varepsilon>0, \exists \delta>0$ st $\sup_n \int_A |X_n| d\mu < \varepsilon$ whenever $\mu(A)<\delta$

But Vitali's Theorem gives simpler conditions.

Proof

To show (1)⇒(2)

(2) $E|X_n - X|^r \longrightarrow 0$ as $n\to\infty$

ie given $\varepsilon>0$ $\exists M_\varepsilon$ st $E|X_n-X|^r < \varepsilon \quad \forall n \geq M_\varepsilon$

let $\varepsilon'>0$ and we will define it later

$$\text{C} = E|X_n-X|^r = \underbrace{E|X_n-X|^r I_{[|X_n-X|>\varepsilon']}}_{\text{A}} + \underbrace{E|X_n-X|^r I_{[|X_n-X|\leq\varepsilon']}}_{\text{B}}$$

Clearly $E|X_n-X|^r I_{[|X_n-X|\leq\varepsilon']} \leq (\varepsilon')^r$

Now we have $X_n \xrightarrow{\mu} X \Rightarrow \exists\, n'$ st $X_{n'} \xrightarrow{a.s} X$

ie $\lim_{n\to\infty} X_{n'} = X = \underline{\lim}\, X_{n'} = \overline{\lim}\, X_n$

Also if $\lim_{n\to\infty} X_{n'} = X$ then $\lim_{n\to\infty} |X_{n'}|^r = |X|^r$

$\therefore \underline{\lim}\, |X_{n'}|^r = |X|^r$

Now $E|X|^r \stackrel{\cdot}{=} E\,\underline{\lim}\,|X_{n'}|^r \leq \underline{\lim}\, E|X_{n'}|^r \leq \sup_n E|X_n|^r < \infty$?

Fatou's lemma by ①

$|X_{n'}|^r$ are u.i & $X \in \mathcal{L}_r$ $\Longrightarrow$ $|X_n - X|^r$ is u.i (claim G)

Since $X_n \xrightarrow{P} X$, given any ε, δ $\exists$ $N(\delta,\varepsilon)$ st $P(|X_n - X| > \varepsilon) < \delta$ $\forall n \geq N(\varepsilon,\delta)$

Now since $\{|X_n - X|^r\}_{n\geq 1}$ is u.i given any ε $\exists$ δ st $\mu(A) < \delta$

then $\sup_n \int_A |X_n - X|^r \, dP < \varepsilon$ {by def of u.i

$\therefore$ for ε' $\exists$ δ' st $\mu(A) < \delta'$ and $\sup_n \int_A |X_n - X|^r \, dP < \varepsilon'$

we also have ε', δ' st $\exists$ $N(\delta', \varepsilon')$ and $P(|X_n - X| > \varepsilon') < \delta'$ $\forall n \geq N(\delta', \varepsilon')$

Now take a fixed $n^* \geq N(\varepsilon', \delta')$, we have $P(|X_{n^*} - X| > \varepsilon') < \delta'$

$\Rightarrow$ $\sup_n \int_{\{|X_{n^*} - X| > \varepsilon'\}} |X_n - X|^r \, dP < \varepsilon'$ [let $A = \{|X_{n^*} - X| > \varepsilon'\}$

Also $\int_{\{|X_{n^*} - X| > \varepsilon'\}} |X_{n^*} - X|^r \, dP \leq \sup_n \int_{\{|X_{n^*} - X| > \varepsilon'\}} |X_n - X|^r \, dP < \varepsilon'$

ie $\int_{\{|X_n - X| > \varepsilon'\}} |X_n - X|^r \, dP < \varepsilon'$ $\forall n \geq N(\varepsilon', \delta')$

ie $E\, |X_n - X|^r \, I_{[|X_n - X| > \varepsilon']} < \varepsilon'$

$\therefore$ © $< \varepsilon' + (\varepsilon')^r$ $\Longrightarrow$ © $< \varepsilon$ [let $\varepsilon = \varepsilon' + (\varepsilon')^r$

THEOREM

Let $S_c = \{ \sum y_j I_{I_j} \mid I_j$ are disjoint finite intervals $\}$

Then $\forall \varepsilon > 0$, $\exists\ Y_\varepsilon \in S_c$ st $\int |X - Y_\varepsilon| d\mu_F < \infty$ for any $X \in \mathcal{L}_1$

NOTE: Any funct^n X in $\mathcal{L}_1$ can be approximated by a piecewise funct^n

eg: Let X be a density funct^n which is in $\mathcal{L}_1$ (w.r.t lebesgue meas λ) then you can use histograms to approximate this density.

Proof Let $X \geq 0$.

Recall Prop 2.2.3 : $\forall \varepsilon > 0$ $\exists$ Z_ε a simple funct^n st

$$|X(\omega) - Z_\varepsilon(\omega)| \leq \varepsilon \qquad \forall \omega \in \Omega.$$

$$Z_\varepsilon = \sum_{i=1}^{n} z_i I_{A_i} \quad \text{st} \quad z_i \in \mathbb{R} \ \& \ \sum_{i=1}^{n} A_i = \mathcal{B}$$

Note: The diff b/w Z_ε & Y_ε is that here A_i are disjoint borel sets and not disjoint finite intervals

Now μ_F is a L-S measure.

Note $\mu(A_i) < \infty$ $\forall i = 1 \ldots n$ for X in $\mathcal{L}_1$

Proof Assume $\exists\ i_0$ st $\mu(A_{i_0}) = \infty$.

Consider
$$\int |X| d\mu \leq \int |X - Z_\varepsilon| \cdot d\mu + \int |Z_\varepsilon| d\mu$$
$$= \int |X - Z_\varepsilon| d\mu + \int \Big|\sum_{i=1}^{n} z_i I_{A_i}\Big| d\mu$$
$$\leq \int |X - Z_\varepsilon| \cdot d\mu + \int \sum |z_i| I_{A_i} \cdot d\mu$$
$$= \int |X - Z_\varepsilon| d\mu + \sum z_i \mu(A_i)$$
$$= \infty$$
which is a $\rightarrow\leftarrow$ since $X \in \mathcal{L}_1$

Note also that $\int |X - Z_\varepsilon| \, d\mu < \infty$

since $\int |X - Z_\varepsilon| \, d\mu \leq \int |X| + \int |Z_\varepsilon|$

$< \infty$

$\left\{ \begin{array}{l} \text{since } X \in \mathcal{L}_1 \\ \& \ \mu(A_i) < \infty \quad \forall A_i \end{array} \right.$

Choose Z_ε st $\int |X - Z_\varepsilon| < \varepsilon/2$

$\left\{ \begin{array}{l} \text{since } \int |X - Z_\varepsilon| \leq M \\ \text{we can rescale so it is} \leq \varepsilon/2 . \end{array} \right.$

(*)

Pg 16 Ex 1.2.3 Halmos approximation lemma

Here we have $\mathcal{B} = \sigma[\mathcal{C}_F]$

$\mathcal{C}_F \supset \{ \sum_n I_n : \text{each } I_n \text{ is intv of type } (a, b] \}$

Also $A_1, A_2 \ldots A_n \in \mathcal{B}$ and $\mu(A_i) < \infty \ \forall A_i$ then

$\exists \ B_1, B_2 \ldots B_n \in \mathcal{C}_F$ st $\mu(A_i \Delta B_i) < \dfrac{\varepsilon}{2n|z_i|}$

where $A_i \Delta B_i = (A_i^c \cap B_i) \cup (B_i^c \cap A_i)$

Now we define $Y_\varepsilon \equiv \sum_{i=1}^{n} z_i I_{B_i}$

Consider $\int |Z_\varepsilon - Y_\varepsilon| \cdot d\mu = \int \left| \sum_{i=1}^{n} z_i (I_{B_i} - I_{A_i}) \right| \cdot d\mu$

$\leq \int \sum_{i=1}^{n} |z_i| |I_{B_i} - I_{A_i}| \, d\mu$

$= \sum |z_i| \int (|I_{B_i} - I_{A_i}|) \cdot d\mu .$

Now $|I_{B_i} - I_{A_i}| = \begin{cases} 1 & \text{if } \omega \in A_i \setminus B_i = A_i \cap B_i^c \\ 1 & \text{if } \omega \in B_i \setminus A_i = A_i^c \cap B_i \\ 0 & \text{if } \omega \in B_i A_i \end{cases}$

$$\int | I_{B_i} - I_{A_i} | \, d\mu = \int_{A_i B_i^c \cup A_i^c B_i} 1 \, d\mu = \mu (A_i B_i^c \cup A_i^c B_i).$$

$$\therefore \int | Z_\varepsilon - Y_\varepsilon | \, d\mu \leq \sum |z_i| \, \mu (A_i \Delta B_i)$$

$$\leq \sum_{i=1}^{n} |z_i| \frac{\varepsilon}{2n|z_i|}$$

$$= \sum_{i=1}^{n} \frac{\varepsilon}{2n}$$

$$= \frac{\varepsilon}{2} \qquad \text{——} (**)$$

Now $\int |X - Y_\varepsilon| \cdot d\mu \leq \int |X - Z_\varepsilon| + \int |Z_\varepsilon - Y_\varepsilon|$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \qquad \text{from } (*) \,\&\, (**)$$

Rewrite $Y_\varepsilon = \sum_{i=1}^{n} z_i I_{B_i} \underset{\substack{\uparrow \\ \text{rewrite}}}{=} \sum_{j=1}^{m} y_j I_{I_j}$ where I_j are disjoint intervals

eg: If $Y_\varepsilon = 3 I_{[0,1] \cup (1,7)} + 2 I_{[0,1] \cup (1,8)}$

$$= 5 I_{[0,1]} + 5 I_{(1,7)} + 2 I_{(7,8)}$$

5.7 → Theorem 5.7

Know def & statements of all modes in this theorem.

Inequalities (Statements)

Pg 51, 52, 53, 54, 55, 56

Proof on pg 58

Theorem 5.8 (Proof in class)

Whole Statement

No theorem 5.6

Inequalities (No proofs) — Minkovski's inequality

Chebyshev's

Markov's

Other ones which were used in class.

Last 2 HW - Proof

Statements & Definition (10 pts)

Sanjay's Problem