II. Convergence in measure

Definition 1 Let $\{X_n\}_n$ be a sequence of a.e. finite measurable functions. Let X be a measurable function with values in $\bar{\mathbb{R}}$. Then

$$X_n \xrightarrow[n\to\infty]{\mu} X \iff \forall\, \varepsilon > 0,\ \mu([|X_n - X| \geq \varepsilon]) \to 0, \text{ for } n \to \infty.$$

Note X is then a.e. finite.

Definition 2

$\{X_n\}_n$ converges mutually in measure, $X_m - X_n \xrightarrow{\mu} 0$

$$\iff \mu([|X_m - X_n| \geq \varepsilon]) \to 0, \text{ as } m \wedge n \to \infty,$$

$\forall\, \varepsilon > 0$.

Look at examples

Connection b/w $\xrightarrow{a.e}$ & $\xrightarrow{\mu}$

Proposition (The limit in measure is a.e. unique).

If $X_n \xrightarrow{\mu} X$ and $X_n \xrightarrow{\mu} \tilde{X}$, then $\tilde{X} = X$ a.e.

Proof: We need to show that

$$\mu([\tilde{X} \neq X]) = 0. \quad //$$

$$0 \leq \mu([\tilde{X} \neq X]) = \mu\Big(\bigcup_k [|X-\tilde{X}| \geq \tfrac{1}{k}]\Big)$$

$$\leq \sum_{k=1}^{\infty} \mu([|X-\tilde{X}| \geq \tfrac{1}{k}]) = 0,$$

hence $\mu([\tilde{X} \neq X]) = 0$, provided that we show that $\mu([|X-\tilde{X}| \geq \varepsilon]) \stackrel{!}{=} 0$, for all $\varepsilon > 0$.

Now:

$$\underline{\underline{0}} \leq \underline{\mu([|X-\tilde{X}| \geq 2\varepsilon])} = \lim_{n\to\infty} \mu([|X-\tilde{X}| \geq 2\varepsilon]) =$$

$$= \lim_{n\to\infty} \mu([|X - X_n + X_n - \tilde{X}| \geq 2\varepsilon]) \leq$$

$$\leq \lim_{n\to\infty} \Big\{ \mu([|X-X_n| > \varepsilon]) + \mu([|X_n - \tilde{X}| > \varepsilon]) \Big\} \underline{\underline{= 0}}$$

q.e.d.

Remark 1) $\xrightarrow{\mu}$ DOES NOT, in general, imply $\xrightarrow{a.e}$

2) If $\mu(\Omega) = \infty$, $\xrightarrow{a.e}$ DOES NOT ~~imply~~ imply $\xrightarrow{\mu}$.

Theorem (The link between $\xrightarrow{a.e}$ and $\xrightarrow{\mu}$).

Let X and $\{X_n\}_n$ be measurable and finite a.e. Then the following results hold:

1) $X_n \xrightarrow{a.e.} X \iff X_m - X_n \xrightarrow{a.e} 0$

2) $X_n \xrightarrow{\mu} X \iff X_m - X_n \xrightarrow{\mu} 0$

3) $\boxed{\text{Let } \mu(\Omega) < \infty.}$ Then

$$X_n \xrightarrow{a.e.} X \implies X_n \xrightarrow{\mu} X.$$

4) (Riesz).

$X_n \xrightarrow{\mu} X \implies \exists \{X_{n_k}\}$ a subsequence of $\{X_n\}$ such that $X_{n_k} \xrightarrow{a.e} X$.

<u>Proof</u> 1) $\leftarrow$ Prop. 3.1 page 29 (Ask)

2) $\leftarrow$ Exercise !

3) We need to show that, if $\mu(\Omega) < \infty$,

$$\forall \varepsilon > 0 \quad \mu([|X_n - X| > \varepsilon]) \xrightarrow[n \to \infty]{} 0$$

But $\mu([|X_n - X| > \varepsilon) \leq \mu\left(\bigcup_{m=n}^{\infty} |X_m - X| \geq \varepsilon\right) \to 0$

(using convergence a.e. and reasoning as in Prop. 3.3 page 30).

4) Let $X_n \xrightarrow[n \to \infty]{\mu} X$.

We construct a subsequence n_K such that

$$X_{n_K} \xrightarrow[K \to \infty]{a.e} X.$$

First recall what $X_n \xrightarrow[n\to\infty]{\mu} X$ means:

$$\forall \varepsilon>0 \quad \lim_{n\to\infty} \mu([|X_n - X| \geq \varepsilon]) = 0 \iff$$

$\iff \forall \varepsilon>0 \quad \exists n_\varepsilon$ s.t. $\forall n \geq n_\varepsilon$ we have.

$$\mu([|X_n - X| \geq \varepsilon]) \leq \varepsilon .$$

$\iff \forall \varepsilon = \frac{1}{2^k} > 0 \; \exists n_K$ s.t. $\forall n \geq n_K$ we have

$$\mu([|X_n - X| \geq \tfrac{1}{2^k}]) < \frac{1}{2^k} .$$

<u>Thus</u> we can choose a sequence $n_K \nearrow$ such that

1) $\mu(A_k) \equiv \mu([|X_{n_k} - X| > \frac{1}{2^k}]) < \frac{1}{2^k}$

2) $\mu([|X_n - X| > \frac{1}{2^k}]) < \frac{1}{2^k}$, for all $n \geq n_k$.

<u>Thus</u>: $A_K \equiv [|X_{n_K} - X| > \frac{1}{2^K}]$

- We need to show that

$$X_{n_K}(\omega) \xrightarrow[K\to\infty]{} X(\omega), \text{ for all } \omega \in C \text{ with } \mu(C^c)=0.$$

- In what follows we identify the convergence set C and show that $\mu(C^c)=0$.
- With $A_K \equiv [|X_{n_K} - X| > \frac{1}{2^K}]$, $K \geq 1$, define

$$B_m \equiv \bigcup_{K=m}^{\infty} A_K.$$

<u>Thus</u> $B_m^c = \bigcap_{K=m}^{\infty} A_K^c = \bigcap_{K=m}^{\infty} \{\omega \in \Omega \mid |X_{n_K}(\omega) - X(\omega)| \leq \frac{1}{2^K}\}$,

<u>Let</u> $C \equiv \bigcup_{m=1}^{\infty} B_m^c = \bigcup_{m=1}^{\infty} \bigcap_{K=m}^{\infty} \{\omega \in \Omega \mid |X_{n_K}(\omega) - X(\omega)| \leq \frac{1}{2^K}\}$.

Let $\omega \in C \Rightarrow \exists\, m \geq 1$ s.t. $|X_{n_K}(\omega) - X(\omega)| \leq \frac{1}{2^K}$, for all $k \geq m$. $\xRightarrow{\text{Since } \{n_K\} \uparrow \text{ in } K}$

$\exists\, n_m \geq 1$ s.t. for all $n_K \geq n_m$: $|X_{n_K}(\omega) - X(\omega)| \leq \frac{1}{2^K} \xrightarrow[K\to\infty]{} 0$

- <u>Thus</u> C is the convergence set.
- $$\mu(C^c) = \mu\left(\bigcap_{m=1}^{\infty} B_m\right) = \mu\left(\bigcap_{m=1}^{\infty} \bigcup_{K=m}^{\infty} A_K\right)$$

$$= \lim_{m\to\infty} \mu\left(\bigcup_{K=m}^{\infty} A_K\right) \leq \lim_{m\to\infty} \sum_{K=m}^{\infty} \mu(A_K)$$

$$= \lim_{m\to\infty} \sum_{K=m}^{\infty} \frac{1}{2^K} = 0$$ (the "rest" of a convergent series converges to zero).

q. e. d.

Induced measures

Assume we have $(\Omega, \mathcal{A}, \mu)$ and $(\Omega', \mathcal{A}')$.
Also, assume we have $X : (\Omega, \mathcal{A}) \to (\Omega', \mathcal{A}')$ which is $\mathcal{A}'$-$\mathcal{A}$ measurable.

Then: X induces a measure on $(\Omega', \mathcal{A}')$.
Denote the induced measure by μ_X. For uniformity, let $\mu_X \equiv \mu'$.

Define $\mu'(A') = \mu(X^{-1}(A'))$, for each $A' \in \mathcal{A}'$.

$\boxed{\mu' \text{ is a measure}}$ //

a) $\mu'(\emptyset) = \mu(X^{-1}(\emptyset)) = \mu(\emptyset) = 0$

b) $\mu'(\sum_n A_n') = \mu(X^{-1}(\sum_n A_n')) =$

$= \mu(\sum_n X^{-1}(A_n')) \underset{\substack{\uparrow \\ \mu \text{ measure}}}{=} \sum_n \mu(X^{-1}(A_n')) =$

$= \sum_n \mu'(A_n')$ //

Also, note that:

$$\mu'(\Omega') = \mu(X^{-1}(\Omega')) = \mu(\Omega),$$ so

Ask Shuva

if ~~$\mu(\Omega) = 1$~~ μ is a prob. measure, the induced measure is also a probability.

- For $X : (\Omega, \mathcal{A}) \to (\bar{\mathbb{R}}, \bar{\mathcal{B}})$ measurable,* we denote by $\mathcal{F}(X) \equiv X^{-1}(\bar{\mathcal{B}})$ which will be called the σ-field generated by X. Notice that it is a σ-field by Prop. 1.2.

- Hence, ~~given~~ in general, given $X : \Omega \to \Omega'$, if we have $(\Omega', \mathcal{A}')$ a field, then we can always induce a σ-field on Ω, namely $\mathcal{F}(X)$.

~~$(\Omega, \mathcal{F}(X))$~~

Then, if $X : (\Omega, \mathcal{F}(X), \mu) \to (\Omega', \mathcal{A}')$, we can always ~~can~~ induce a measure μ_X on $\mathcal{A}$!

Proposition 2.5 (The form of an $\mathcal{F}(X)$ measurable function).

Let $X : (\Omega, \mathcal{A}) \to (\mathbb{R}, \mathcal{B})$ measurable.

Let $Y : (\Omega, \mathcal{F}(X)) \to (\mathbb{R}, \mathcal{B})$, $\mathcal{F}(X)\ (\subset \mathcal{A})$ measurable.

Then there exists $g : (\mathbb{R}, \mathcal{B}) \to (\mathbb{R}, \mathcal{B})$, measurable, such that $Y = g(X)$.

Proof Recall that

$$\mathcal{F}(X) \equiv X^{-1}(\mathcal{B}) = \{ X^{-1}(B),\ B \in \mathcal{B} \}.$$

Step 1 Suppose $Y = 1_D$, for some $D \in \mathcal{F}(X)$.

Then, Y is $\underset{\mathcal{F}(X)}{\wedge}$ measurable.

Notice now that:

$$Y(\omega) = 1_D(\omega) = 1_{X^{-1}(B)}(\omega) = \begin{cases} 1, & \text{if } \omega \in X^{-1}(B) \\ 0, & \text{if } \omega \notin X^{-1}(B) \end{cases}$$

$$= \begin{cases} 1, & \text{if } \omega \in \{\omega \mid X(\omega) \in B\} \\ 0, & \text{otherwise} \end{cases}$$

$$= 1_B(X(\omega)) \equiv g(X(\omega))$$

for some $B \in \mathcal{B}$

Hence $Y = g \circ X$ and note that since $g(y) = 1_B(y)$, $B \in \mathcal{B}$, then g is measurable (Proposition 2.3).

Step 2 Suppose now that Y is a simple function

$$Y = \sum_{i=1}^{m} d_i 1_{D_i} \underset{\text{as before}}{=} \sum_{i=1}^{m} d_i 1_{B_i}(X) \equiv g(X),$$

and again g is measurable, since all B_i; $i = 1, \dots, m$ are in $\mathcal{B}$ (again Prop. 2.3).

Step 3 Suppose that $Y \geq 0$ is $\mathcal{F}(X)$ measurable.
Then, by Prop 2.3 (g) there exist $\uparrow$ simple $\mathcal{F}(X)$ measurable functions Y_n such that

$$Y = \lim_n Y_n$$

For each Y_n use Step 2 above to conclude that $Y_n = g_n(X)$, with $g_n \uparrow$ simple $\mathcal{B}$ measurable.
Thus $Y = \lim_n Y_n = \lim_n g_n(X) \equiv g(X)$ and use proposition 2.2 to conclude that g is measurable.
For general $Y = Y^+ - Y^-$, construct $g = g^+ - g^-$.

Q.ed.

Lecture 8

Probability, random variables and convergence in law

- Let $X : \underbrace{(\Omega, \mathcal{A}, P)}_{\text{A probability space. } P(\Omega)=1.} \longrightarrow (\mathbb{R}, \mathcal{B})$.

- We call X a r.v.

Define $F_X(x) = P(X \leq x)$ the d.f.

Notice that $F \uparrow$, right continuous, $F(-\infty)=0$, $F(\infty)=1$.

- <u>IMPORTANT</u>: In practice, we see <u>values</u> of X, not X itself. These are <u>numbers</u> in $\mathbb{R}$. Thus, it is important to give (have) a measure on $\mathcal{B}$, since most of the action we are interested in happens there.

• Recall that X induces a measure (probability in this case) on $\mathcal{B}$:

$$P_X(B) = P \circ X^{-1}(B) = P([X \in B]), \text{ for all } B \in \mathcal{B}.$$

It is enough to give a description of P_X on sets of the type $(-\infty, x]$, $x \in \mathbb{R}$.

<u>Hence</u>: $P_X((-\infty, x]) = \mathbb{P}(X \leq x) = F(x)$. ①

<u>Very important</u>: Relationship ① says that in order to compute probabilities of events related to X we need to specify either P_X or F. Of course, once any of them is specified, the other one is also known. In general, it is simpler to define functions, than to define measures directly.

- We use the notation $\boxed{X \sim F}$ to indicate that the induced measure P_X is related to F as in (1).

Definition: Let $\{X_n\}_n$ be a sequence of random variables, each with df F_n. Let $X_0 \sim F_0$. We say that X_n converges in distribution to X_0 (or that X_n converges in law to X_0) if (2) $F_n(x) \xrightarrow[n\to\infty]{} F_0(x)$, at each continuity point of F.

Note that (2) means

$$P(X_n \leq x) \xrightarrow[n\to\infty]{} P(X_0 \leq x).$$

- Notation: $\boxed{X_n \xrightarrow{d} X_0}$

$$F_n \xrightarrow{d} F_0.$$

$$\mathcal{L}(X_n) \longrightarrow \mathcal{L}(X_0)$$

__Proposition 4.1__ Let $X_n \sim F_n$ and $X \sim F$. Then:

$$X_n \xrightarrow{P} X \implies X_n \xrightarrow{d} X.$$

is it only for P or any meas?

Cause a σ-S meas also has a corresponding dist^n funct^n

__Proof__ We need to show that $F_n(t) \xrightarrow[n\to\infty]{} F(t)$,

for all continuity points of F.

1) $$F_n(t) = P(X_n \le t) \overset{?}{\le} P(X_n + X - X \le t) =$$

$$= P(X_n - X + X \le t + \varepsilon - \varepsilon) \le$$

$$\le P(X \le t+\varepsilon) + P(-X_n + X \ge \varepsilon)$$

Since $X \ge t+\varepsilon$ __and__ $X_n - X \ge -\varepsilon$ $\Big\} \implies X_n \ge t$, and then reverse.

$\{X \ge t+\varepsilon \text{ and } X_n - X \ge -\varepsilon\} \subseteq \{X_n \ge t\}$

$P(\quad) \le P(\quad)$

$$\le P(X \le t+\varepsilon) + P(|X_n - X| \ge \varepsilon).$$

By going to a larger event

$\leq F(t+\varepsilon)+\varepsilon$, for all $n \geq n_\varepsilon$ since $X_n \xrightarrow{P} X$.

<u>Thus</u> $\forall \varepsilon > 0 \ \exists n_\varepsilon$ s.t. $\forall n \geq n_\varepsilon$:

$$F_n(t) \leq F(t+\varepsilon)+\varepsilon . \quad (1)$$

2) $F_n(t) = P(X_n \leq t) \geq P(X \leq t-\varepsilon \text{ and } |X_n - X| \leq \varepsilon)$

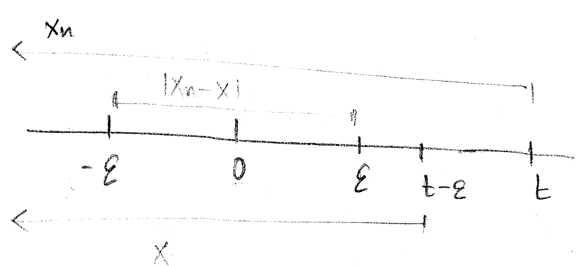
since, again, $\{X \leq t-\varepsilon \text{ and } |X_n - X| \leq \varepsilon\} \subseteq \{X_n \leq t\}$

$$\equiv P(AB) \geq P(A) - P(A \cap B^c) \geq$$

$$\geq P(A) - P(B^c) = P(X \leq t-\varepsilon) - \underline{P(|X_n - X| \geq \varepsilon)}$$

Thus, using again that $X_n \xrightarrow{P} X$, we have $\forall \varepsilon > 0 \ \exists n'_\varepsilon$ st $\forall n \geq n'_\varepsilon$:

$$F_n(t) \geq F(t-\varepsilon)-\varepsilon \quad (2)$$



From ① $\Rightarrow$ $\overline{\lim}\, F_n(t) \leq F(t+\varepsilon)+\varepsilon$
From ② $\Rightarrow$ $\underline{\lim}\, F_n(t) \geq F(t-\varepsilon)-\varepsilon$
$\Rightarrow$

$$F(t-\varepsilon)-\varepsilon \leq \underline{\lim}\, F_n(t) \leq \overline{\lim}\, F_n(t) \leq F(t+\varepsilon)+\varepsilon$$

If t is a continuity point of F, by letting $\varepsilon \to 0$ we have:

$$\underline{\lim}\, F_n(t) = \overline{\lim}\, F_n(t) \equiv \lim_{n\to\infty} F_n(t) = F(t) .$$

2 ed.

<u>Theorem</u> (Slutsky) $\longrightarrow$ VERY IMPORTANT
used in limit theory!

If
$$\left.\begin{aligned} X_n &\xrightarrow{d} X \\ Y_n &\xrightarrow{P} a \\ Z_n &\xrightarrow{P} b \end{aligned}\right\} \Rightarrow Y_n \cdot X_n + Z_n \xrightarrow{d} aX+b$$

for any constants a and b.

Remark 1 If for a non measurable function X, we have $X = Y$ a.e, for Y measurable, then $\int X \, d\mu = \int Y \, d\mu$.

Remark 2 • In definition 1, if X is measurable and $\int X \, d\mu < \infty$, then X is called ~~measurable~~ integrable.

• For any $A \in \mathcal{A}$, $\int_A X \, d\mu \equiv \int X \cdot 1_A \, d\mu$.

Proposition 1 $\int X \, d\mu$ (given by Definition 1) is well defined.

Proof: You need to show that if

$$X = \sum_1^n x_i 1_{A_i} = \sum_{j=1}^m y_j 1_{B_j}, \text{ then}$$

$\int X \, d\mu$ as defined by 1) does not ~~defined~~ depend on the particular representation of X. That will mean that $\int X d\mu$ is well defined for simple functions, and hence for the rest.

See Step 1 in the Proof of Prop. 1.1. page 38.

Proposition 2 Suppose that $\int X d\mu + \int Y d\mu$ is a well defined number in $[-\infty, \infty]$. Then:

a) $\int (X+Y) d\mu = \int X d\mu + \int Y d\mu$

a') $\int cX d\mu = c \int X d\mu$, for any $c \in \mathbb{R}$.

b) $X \geq 0 \Longrightarrow \int X d\mu \geq 0$.

Proof a') and b) $\longrightarrow$ Exercise!

To prove a) we employ the usual strategy:

- • First we show that it holds for simple functions.
- •• Then, we show that it holds for $X \geq 0$ and $Y \geq 0$. (After the proof of the next Th).
- ••• After we proved the first two, we write $X = X^+ - X^-$, $Y = Y^+ - Y^-$ and use •, •• and a') to draw the conclusion.

Prove $\odot_{II}$.

Let $X = \sum_{i=1}^{m} x_i 1_{A_i}$, $Y = \sum_{j=1}^{n} y_j 1_{B_j}$.

Then $X + Y = \sum_{i=1}^{m} \sum_{j=1}^{n} (x_i + y_j) 1_{A_i \cdot B_j}$.

Hence:

$$\int (X+Y)\, d\mu = \sum_{i=1}^{m} \sum_{j=1}^{n} (x_i + y_j)\, \mu(A_i B_j) =$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} x_i\, \mu(A_i B_j) + \sum_{i=1}^{m} \sum_{j=1}^{n} y_j\, \mu(A_i B_j) =$$

$$= \sum_{i=1}^{m} x_i \sum_{j=1}^{n} \mu(A_i B_j) + \sum_{j=1}^{n} y_j \sum_{i=1}^{m} \mu(A_i B_j) =$$

$$= \sum_{i=1}^{m} x_i\, \mu(A_i) + \sum_{j=1}^{n} y_j\, \mu(B_j),$$ since

$\{A_i\}_i$ and $\{B_j\}_j$ are partitions of Ω

$$= \int X\, d\mu + \int Y\, d\mu.$$

Q.e.d.

<u>Theorem 1</u> (MCT, the monotone convergence theorem)

Let $\{X_n\}_n$ be an increasing sequence of measurable functions $X_n \geq 0$ such that

$X_n \uparrow X$ a.e.

<u>Then</u>: $0 \leq \int X_n d\mu \uparrow \int X d\mu$.

<u>Proof</u> • Firstly, recall Remark 1. Thus, by redefining on null sets if needed, we can assume that $X_n \longrightarrow X$ everywhere and so X itself is measurable.

• Then, by the monotonicity of the integral ((b) of Prop. 2) and the fact that $X_n \uparrow$ we also have that $\int X_n \uparrow$. Thus, $\int X_n$ is an increasing sequence of positive numbers, so it must have a limit in $[0, \infty]$.

Let $a \equiv \lim_{n \to \infty} \int X_n d\mu$.

Invoking again monotonicity, s[redacted]ince $\leq$ is the limit of an increasing [redacted]sequence we also have $\int X_n \, d\mu \leq \int X \, d\mu$.

- **So**: $\lim_{n \to \infty} \int X_n \, d\mu \equiv a \leq \int$ u.

In what follows we show t[redacted]at we also have $\int X \, d\mu \leq a$, which will imply that $\lim_{n \to \infty} \int X_n \, d\mu = a = \int X \, d$

<u>Show that $\int X \, d\mu \leq a$</u>

Recall the definition of $\int X \, d\mu$, [redacted]or X:

$$\int X \, d\mu \equiv \sup \{ \int Y d\mu : 0 \leq Y \leq X \text{ and is a simple function}$$

Let $0 \leq Y \leq X$, Y simple be ar[redacted]itrary.
If we can prove that $\int Y d\mu \leq$ [redacted]a for such Y, then the sup will have the sa[redacted]e property, and we are done!

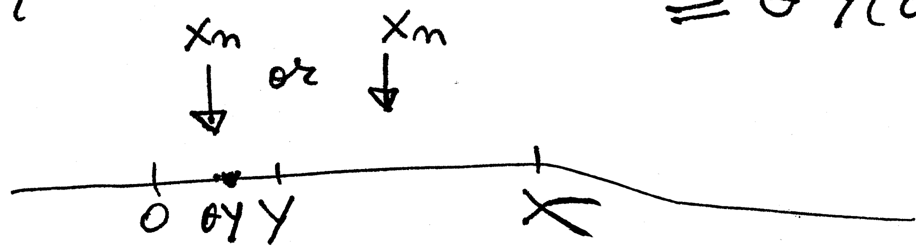
Thus, we only need to show that

$$\underline{\int Y d\mu \le a = \underset{\parallel}{\lim_{n\to\infty}} \int X_n d\mu,} \text{ — for simple } Y\text{'s } 0 \le Y \le X.$$

Let $0<\theta<1$ be arbitrary, fixed.

Consider $A_n \equiv \{\omega \in \Omega \mid X_n(\omega) \ge \theta Y(\omega)\}$.

Recall that we have



Note that $\bigcup_n^{\infty} A_n = \Omega$ and $\{A_n\}_n \nearrow$, since $\{X_n\}_n \nearrow$

"$\subseteq$" Trivial. ~~since X_n, Y measurable~~.

"$\supseteq$" Let $\omega \in \Omega$. Consider two cases:

a) $\underline{\omega \in \Omega}$ for which ~~and~~ $X(\omega) > 0$.

Then, since $\lim_{n\to\infty} X_n(\omega) = X(\omega) \ge Y(\omega) > \theta Y(\omega) \Rightarrow$

$\Rightarrow \exists n_\varepsilon$ s.t. $\forall n > n_\varepsilon \Rightarrow X_n(\omega) > \theta Y(\omega) \Rightarrow$

$\Rightarrow \omega \in$ (some) A_n's $\Rightarrow \omega \in \bigcup_n^{\infty} A_n$.

b) $\omega \in \Omega$ s.t. $X(\omega) = 0$. Then $Y(\omega) = 0$ and $\omega \in A_n$, for all n.

Then, first note that:

(10) $\theta \int Y \cdot 1_{A_n} \underset{\substack{\uparrow \\ \text{Proposition 2}}}{=} \int \theta Y \cdot 1_{A_n} \underset{\substack{\uparrow \\ \text{on } A_n}}{\leq} \int X_n \cdot 1_{A_n} \underset{\substack{\uparrow \\ \text{Bounding the indicator by 1}}}{\leq} \int X_n$

Since $\{\int X_n\} \nearrow$ and a is its limit

Now, recall that Y is a simple function, say

$Y = \sum_{j=1}^{m} y_j \cdot 1_{B_j}$. Then

$Y \cdot 1_{A_n} = \sum_{j=1}^{m} y_j \cdot 1_{B_j \cdot A_n}$ and so

(11) $\int Y 1_{A_n} d\mu = \sum_{j=1}^{m} y_j \mu(B_j \cdot A_n) \underset{\substack{\uparrow \\ \text{Prop 1.2.} \\ \text{and } A_n \uparrow \Omega}}{\nearrow} \sum_{j=1}^{m} y_j \cdot \mu(B_j) =$

$= \int Y d\mu.$

By (10) and (11) we thus have:

$\theta \int Y d\mu \leq a$, for all $0 < \theta < 1$. Take now $\theta \uparrow 1$,

thus $\int Y d\mu \leq a$, which is what we needed

to prove.

q.e.d.

Corollary (Proof of ⊙⊙, page 3) The linearity of the integral for general $X \geq 0$ and $Y \geq 0$.

Proof

Let X_n, Y_n be as defined in display (10) page 26.

Thus, X_n, Y_n measurable, simple, and $X_n \uparrow X$, $Y_n \uparrow Y$.

Then $X_n + Y_n \uparrow X + Y$. (12). Thus:

$$\int X + \int Y \underset{\substack{\uparrow \\ \text{MCT for } X \text{ and } Y\text{, respect.}}}{=} \lim_{n\to\infty} \int X_n + \lim_{n\to\infty} \int Y_n =$$

$$= \lim_{n\to\infty} \left(\int X_n + \int Y_n \right) \underset{\substack{\uparrow \\ \text{By simple fctns' linearity}}}{=} \lim_{n\to\infty} \int (X_n + Y_n) =$$

$$\underset{\substack{\uparrow \\ \text{MCT}}}{=} \int X+Y. \quad //$$

$\leftarrow$ If $\lim \int \neq \int \lim$ then wh

Theorem (Fatou's lemma).

For general $\{X_n\}_n$ measurable we have

$$\int \underline{\lim}\, X_n \, d\mu \leq \underline{\lim} \int X_n \, d\mu \text{ , if } X_n \geq 0 \text{ a.e. for all.}$$

Proof: Recall that, for each ω:

$$\underline{\lim}\, X_n(\omega) = \lim_{n\to\infty} \underbrace{\left(\inf_{k \geq n} X_k(\omega) \right)}_{= Y_n} \overset{\uparrow}{=} \text{ since } Y_n \uparrow$$

$$= \sup_{n \geq 1} \left(\inf_{k \geq n} X_k(\omega) \right).$$

Then: $X_n \geq Y_n \equiv \inf_{K \geq n} X_K \uparrow \underline{\lim}\, X_n$

Thus $Y_n \uparrow \underline{\lim}\, X_n$, and we can apply MCT:

$$\lim_{n\to\infty} \int Y_n = \int \lim_{n\to\infty} Y_n .$$ So:

$$\int \underline{\lim}\, X_n = \int \lim Y_n = \lim \int Y_n = \underline{\lim} \int Y_n \leq$$

$$\leq \underline{\lim} \int X_n, \text{ since } Y_n \leq X_n$$

q.ed.

Theorem (Very important!) **DCT: The dominated convergence theorem.**

(NB I will suppress a.e. where appropriate)

Suppose now that $|X_n| \overset{a.e}{\leq} Y$ for all n, for some function $Y \in \mathcal{L}_1 \equiv \{g \mid \int |g| d\mu < \infty\}$.

Suppose that either of the below hold.

(i) $X_n \xrightarrow{a.e.} X$

or

(ii) $X_n \xrightarrow{\mu} X$

Then: $\int |X_n - X| d\mu \underset{n \to \infty}{\longrightarrow} 0$

(We sometimes denote this as $X_n \xrightarrow[\mathcal{L}_1]{} X$).

Remark: We always have $|X_n| \leq \sup_{n \geq 1} |X_n|$. Then, we can take $Y = \sup_{n \geq 1} |X_n|$, provided that we can prove that $Y \in \mathcal{L}_1$.

eg: $\int_0^\infty \frac{1}{x} dx = \infty \notin \mathcal{L}_1$

$\int_0^\infty \frac{1}{x^2} dx < \infty \in \mathcal{L}_1$

Lecture 10

1. The dominated convergence theorem (DCT).

Proof a) We first show that

$$X_n \xrightarrow{a.e.} X \implies \int |X_n - X| d\mu \xrightarrow[n\to\infty]{} 0 .$$

- Let $Z_n \equiv |X_n - X| \geq 0$.

Since $X_n \xrightarrow{a.e.} X$, then $Z_n \xrightarrow{a.e.} 0. \implies$

$$\underline{\lim}\, Z_n = 0 .$$

By Fatou's Lemma: $\int \underline{\lim}\, Z_n \, d\mu \leq \underline{\lim} \int |Z_n| d\mu$.

Thus $\underline{\lim} \int |Z_n| d\mu = \underline{\lim} \int |X_n - X| d\mu \geq \int 0 = 0$,

that is $\underline{\lim} \int |X_n - X| d\mu \geq 0$.

- We thus have:

$$0 \leq \underline{\lim} \int |X_n - X| d\mu \underset{\substack{\uparrow \\ \text{always true}}}{\leq} \overline{\lim} \int |X_n - X| d\mu .$$

The proof will be completed by showing that

$$\overline{\lim} \int |X_n - X| \, d\mu \leq 0.$$ //

- By the assumption of the theorem we have that,

a.e., $|X_n| \leq Y$; $Y \in \mathcal{L}_1$.

<u>Then</u> $0 \leq Z_n = |X_n - X| \leq |X_n| + |X| \leq 2Y$ a.e.,

(If $|X_n| \leq Y$ and $X_n \xrightarrow{a.e.} X \Rightarrow |X| \leq Y$ a.e.).

Then: $Z_n - 2Y \leq 0 \iff \underbrace{2Y - Z_n}_{U_n} \geq 0$ a.e.

Applying Fatou's Lemma to $\{U_n\}_n$ we obtain:

$$\int \underline{\lim}\, U_n \leq \underline{\lim} \int U_n$$

$$\Updownarrow$$

$$\int \underline{\lim}\,(2Y - Z_n) \leq \underline{\lim} \left[\int 2Y - \int Z_n \right].$$

$$\Updownarrow \quad Z_n \xrightarrow{a.e.} 0.$$

$$\int 2Y \leq \int 2Y + \underline{\lim} \left[- \int Z_n \right].$$

$\Updownarrow$ $\leftarrow$ By cancelling out $\int 2Y < \infty$
Here is where we use the hyp
that $Y \in \mathcal{L}_1$.

$$\underline{\lim}\left[-\int Z_n\right] \geq 0$$

$$\Updownarrow$$

$$-\overline{\lim}\int Z_n \geq 0 \iff \overline{\lim}\int |Z_n| \leq 0.$$

q.e.d.

b) We show now that

$$X_n \xrightarrow{\mu} X \implies \int |X_n - X| \, d\mu \xrightarrow[n \to \infty]{} 0.$$

The dominated convergence theorem (DCT)

$$X_n \xrightarrow{\mu} X \implies \int |X_n - X| \, d\mu \xrightarrow[n \to \infty]{} 0.$$

<u>Proof</u>: Let $Z_n = |X_n - X| \geq 0$.

- Assume $X_n \xrightarrow{\mu} X$. Then $Z_n \xrightarrow{\mu} 0$. (1)

- Let $a \equiv \overline{\lim} \int Z_n \geq 0$. (2).

- We always have $\underline{\lim} \int Z_n \leq \overline{\lim} \int Z_n$

  Since $Z_n \geq 0 \implies \int Z_n \geq 0 \implies \underline{\lim} \int Z_n \geq 0$

- Thus, trivially :

$$0 \leq \underline{\lim} \int Z_n \leq \overline{\lim} \int Z_n = a, \quad \text{with } a \geq 0.$$

The proof of the result reduces thus to showing that $a = 0$.

- Now, since $\overline{\lim} \int Z_n = a$ we can always find a subsequence n' such that

$$③ \quad \lim_{n' \to \infty} \int Z_{n'} = a .$$

- Since, by the hypothesis of the theorem, we have $Z_n \xrightarrow{\mu} 0$ then

$$Z_{n'} \xrightarrow{\mu} 0$$ for <u>all</u> subsequences of Z_n, so in particular for the $\{Z_{n'}\}_{n'}$ that satisfies ③.

Now, by Theorem 2.3.1 page 31, if $Z_n \xrightarrow{\mu} 0$ then any subsequence $Z_{n'}$ has a further subsequence $Z_{n''}$ such that

$$Z_{n''} \xrightarrow{a.e} 0$$

To summarize up to here:

- We have $\int Z_{n'} \xrightarrow[n' \to \infty]{} a$ <u>and</u>

  $Z_{n''} \xrightarrow{a.e} 0$, where $Z_{n''}$ is a subsequence of $Z_{n'}$.

- Since $Z_{n''}$ is a subsequence of $Z_{n'}$, then we al

  $$\int Z_{n''} \xrightarrow[n'' \to \infty]{} a.$$

<u>Thus</u> $\begin{cases} Z_{n''} \xrightarrow{a.e} 0 \\ \int Z_{n''} \to a \quad \boxed{4} \end{cases}$

But, by the first part of the DCT ~~we~~ have that $Z_{n''} \xrightarrow{a.e} 0 \Longrightarrow \int Z_{n''} \to 0 \quad \boxed{5}$

Then $\boxed{4}$ and $\boxed{5}$ $\Longrightarrow$ $a = 0$ which is what we wanted to show.

q.e.d.

Corollary

1) $$\int |X_n - X| \, d\mu \xrightarrow[n\to\infty]{} 0 \implies \int X_n \, d\mu \xrightarrow[n\to\infty]{} \int X \, d\mu$$

2) If $\int |X_n - X| \, d\mu \to 0 \implies \sup_A \left| \int_A X_n \, d\mu - \int_A X \, d\mu \right| \xrightarrow[n\to\infty]{}$

Proof: 1) ~~Just~~ note that:

$$\left| \int X_n - \int X \, d\mu \right| \leq \int |X_n - X| \, d\mu \,. \qquad \left\{ \because \left|\int \cdot\right| \leq \int |\cdot| \right.$$

2) Note that:

$$\left| \int_A X_n - \int_A X \right| = \left| \int_A (X_n - X) \right|$$

$$= \left| \int (X_n - X) \mathbb{1}_A \right|$$

$$\leq \left| \int (X_n - X) \right| \text{, for all } A.$$

q.e.d.

Theorem 3

If $X_n \geq 0$ a.e., for all n, then

$$\int \sum_{n=1}^{\infty} X_n \, d\mu = \sum_{n=1}^{\infty} \int X_n \, d\mu .$$

Proof Apply MCT to $Z_n \equiv \sum_{k=1}^{n} X_k \uparrow$.

Theorem 4 (The absolute continuity of the integral).

Let $X \in \mathcal{L}_1$. Then, for any A s.t. $\mu(A) \to 0$

we have $\int_A |X| \, d\mu \to 0$.

Proof $\int |X| \, d\mu = \int |X| \, 1_{[|X|>n]} + \int |X| \, 1_{[|X| \leq n]}$.

Notice that, by MCT, we have

$\int |X| \, 1_{[|X| \leq n]} \nearrow \int |X|$, since $[|X| \leq n] \uparrow \Omega$.

Then $\int |X| \, 1_{[|X|>n]} \to 0$, or.

$\forall \varepsilon > 0 \quad \exists \ N \equiv N_\varepsilon$ s.t. $\forall n > N$ we have

$$\int |X| \cdot 1_{[|X| > n]} \leq \frac{\varepsilon}{2} .$$

<u>Thus</u> :

$$\int_A |X| \leq \int_A |X| \, 1_{[|X| \leq N]} + \int_A |X| \, 1_{[|X| > N]} \leq$$

$$\leq \int_A |X| \, 1_{[|X| \leq N]} + \int |X| \, 1_{[|X| > N]} .$$

$$= \int_{A \cap [|X| \leq N]} |X| + \frac{\varepsilon}{2} \leq N \mu (A \cap [|X| \leq N]) + \frac{\varepsilon}{2}$$

$$\leq N \cdot \mu(A) + \frac{\varepsilon}{2} . \leq N \cdot \frac{\varepsilon}{2N} + \frac{\varepsilon}{2} = \varepsilon$$

where we used that $\mu(A) \to 0$, so, w log, we can assume $\mu(A) \leq \frac{\varepsilon}{2N}$.

qed

[No Proof]

Theorem 5 (The Th. of the unconscious Statistician.)

Let $X : (\Omega, \mathcal{A}, \mu) \to (\Omega', \mathcal{A}')$ be a measurable function.

Let $g : (\Omega', \mathcal{A}') \to (\bar{\mathbb{R}}, \bar{\mathcal{B}})$ be a measurable function.

Let μ_X be the induced measure on $\mathcal{A}'$, ($\mu_X(A') = \mu(X^{-1}(A'))$, for $A' \in \mathcal{A}'$).

Then:

1) μ_X determines the induced measure $\mu_{g(X)}$ on $(\bar{\mathbb{R}}, \bar{\mathcal{B}})$.

2) $$\int_{X^{-1}(A')} g(X(\omega))\, d\mu(\omega) = \int_{A'} g(x)\, d\mu_X(x), \text{ for all } A' \in \mathcal{A}'.$$

<u>Remark</u> Thus, if X is a r.v.:

$X : (\Omega, \mathcal{A}, P) \longrightarrow (\mathbb{R}, \mathcal{B})$, we first have

$X : (\Omega, \mathcal{A}, P) \longrightarrow (\mathbb{R}, \mathcal{B}, P_X)$.

Then, for any measurable

$g : (\mathbb{R}, \mathcal{B}) \longrightarrow (\bar{\mathbb{R}}, \bar{\mathcal{B}})$, we have (we can

consider):

$g : (\mathbb{R}, \mathcal{B}, P_X) \longrightarrow (\bar{\mathbb{R}}, \bar{\mathcal{B}}, P_{g(X)})$ and, by t[he] theorem, w[e]

$$\int_{X^{-1}(B)} g(X(\omega))\, dP(\omega) = \int_B g(x)\, dP_X(x), \text{ for all } B \in \mathcal{B}.$$

<u>Thus</u>: The only thing we need to know is P_X (hence, we don't need to deduce $P_{g(X)}$, and we don't need to specify the "abstract" P).

<u>Proof</u>. Since X and g are measurable, so is $g \circ X \equiv g(X)$. Then:

$$\mu_{g(X)}(B) \underset{\uparrow}{=} \mu(g(X) \in B) = \mu\left((g \circ X)^{-1}(B)\right)$$

By the def of an induced measure

$$\begin{tikzcd}
(\Omega, \mathcal{A}, \mu) \arrow[r, "X"] \arrow[dr, "g(X)"'] & (\Omega', \mathcal{A}', \mu_X) \arrow[d, "g"] \\
& (\overline{\mathbb{R}}, \overline{\mathcal{B}}, \mu_{g(X)})
\end{tikzcd}$$

$$= \mu\left(X^{-1} \circ g^{-1}(B)\right) = \mu\left(X^{-1}(\underbrace{g^{-1}(B)}_{\in \mathcal{A}'})\right) =$$

$$= \mu_X\left(g^{-1}(B)\right).$$

Hence, for any $B \in \overline{\mathcal{B}}$, $\mu_{g(X)}(B) = \mu_X(g^{-1}(B))$, thus the measure $\mu_{g(X)}$ is known, if μ_X is known.

Prove now that, indeed:

$$\int_{X^{-1}(A')} g(X(\omega))\, d\mu(\omega) = \int_{A'} g(x)\, d\mu_X(x), \text{ for any } A' \in \mathcal{A}'.$$

<u>Step 1</u>: $g = 1_{A'}$. Then

- $\int 1_{A'}(X)\, d\mu = ?$;

$$1_{A'}(X) = \begin{cases} 1, & \text{for } \omega \text{ s.t. } X(\omega) \in A' \\ 0, & \text{otherwise} \end{cases} =$$

$$= \begin{cases} 1, & \text{on } X^{-1}(A') \\ 0, & \text{otherwise} \end{cases} = 1_{X^{-1}(A')}.$$

<u>Hence</u>: $\int 1_{A'}(X)\, d\mu = \int 1_{X^{-1}(A')}\, d\mu \underset{\uparrow}{=} \mu(X^{-1}(A')) =$

definition of μ

$$= \mu_X(A') \underset{\uparrow}{=} \int 1_{A'}\, d\mu_X$$

definition of μ_X

Or:

$$\int_{X^{-1}(A')} d\mu(\omega) = \int_{A'} d\mu_X.$$

Step 2

If $g = \sum_{i=1}^{n} c_i \mathbb{1}_{A_i'}$, with $\sum A_i' = \Omega$, $A_i' \in \mathcal{A}'$,
then by an identical reasoning we get

$$\int g(X)\,d\mu = \int g\,d\mu_X .$$

Step 3. Let $g \geq 0$. Let $g_n \geq 0$ be simple with $g_n \uparrow g$. Then:

$$\int g(X)\,d\mu = \lim_{n\to\infty} \int g_n(X)\,d\mu = \lim_{n\to\infty} \int g_n\,d\mu_X$$

By MCT, since $g_n(X) \uparrow g(X)$ By Step 2

$$= \int g\,d\mu_X$$

Again by MCT, applied now to g_n.

Step 4. Let g be measurable, arbitrary.
Assume that either $\int g(x)^+ d\mu$ or $\int g(x)^- d\mu$ is finite.

Recall that for any g we have $g = g^+ - g^-$.

Also, define $g(x)^+ = g^+(x)$ and
$g(x)^- = g^-(x)$.

Then:

$$\int g(x)\,d\mu = \int g(x)^+ d\mu - \int g(x)^- d\mu =$$

$$= \int g^+(x)\,d\mu - \int g^-(x)\,d\mu =$$

$$\underset{\text{Step 3}}{=} \int g^+ d\mu_X - \int g^- d\mu_X = \int g\,d\mu_X.$$

Qed

Definition

Let $(\mathbb{R}, \hat{\mathcal{B}}_\mu, \mu)$ denote the Lebesgue-Stieltjes measure that has been completed.

If g is $\hat{\mathcal{B}}_\mu$ measurable, then

$\int g \, d\mu$ is called the L-S integral of g.

- Recall the correspondence theorem between a L-S measure μ and a generalized df F.

Then, we also use the <u>notation</u> $\int g \, dF$.

- Also:

$$\int_a^b g \, d\mu \equiv \int_a^b g \, dF \equiv \int_{(a,b]} g \, dF = \int g \, 1_{(a,b]} \, dF .$$

<u>Theorem</u> (The LS and the RS integrals coincide for continuous functions).

Let g be continuous on $[a,b]$. Let F be the cdf corresponding to μ.

Then $\int_a^b g\,d\mu$ (which is a LS integral) = the Riemann-Stieltjes integral $\equiv \int_a^b g\,dF$ of the R-S integral.

<u>Proof</u>: Recall the definition of the R-S integral.

Let $a \equiv x_{n0} < x_{n1} < \ldots < x_{nk} < \ldots < x_{nn} \equiv b$

be a partition of $[a,b]$ such that

$$\text{mesh}_n \equiv \max_{1 \le k \le n} (x_{nk} - x_{nk-1}) \to 0$$

For each $k = \overline{1,n}$ let x_{nk}^* such that

$$x_{n,k-1} < x_{nk}^* \le x_{nk}.$$

Since g is continuous on the compact set $[a,b]$, g is uniformly continuous. Then:

$$g_n \equiv \sum_{k=1}^{n} g(x_{nk}^*) 1_{(x_{n,k-1}, x_{nk}]} \to g \text{ uniformly over } [a,b].$$

Notice that $\exists M$ such that $|g_n(x)| \leq M$, for all $x \in [a,b]$, since we can always bound the indicators of $[a,b]$ by 1 and g is continuous on $[a,b]$, hence bounded.

Then, we can apply the DCT, with dominating function the constant M, to obtain:

-S integral

$$\overset{}{=} \int_a^b g\, d\mu \overset{\text{DCT}}{=} \lim_{n\to\infty} \int_a^b g_n\, d\mu \overset{\text{Definition of } g_n}{=}$$

$$= \lim_{n\to\infty} \int_a^b \left(\sum_{k=1}^{n} g(x_{nk}^*)\, 1_{(x_{n,k-1},\, x_{nk}]} \right) d\mu$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} g(x_{nk}^*)\, \mu(x_{n,k-1}, x_{nk}] =$$

$$\underset{\substack{\text{Correspondence Th} \\ \mu \leftrightarrow F}}{=} \lim_{n\to\infty} \sum_{k=1}^{n} g(x_{nk}^*)\, F(x_{n,k-1}, x_{n,k}] =$$

$$\underset{\text{By the def of a R-S integral}}{=} \lim_{n\to\infty} \left(\text{R-S sums for } \int_a^b g\,dF \right) = \text{the R-S integral.}$$

$$= \int_a^b g\, dF.$$

Qed