STA 5446 – Hwk 1, 2005

Due Tuesday, September 13, 2005.

1. Show that the family of <u>all subsets</u> 2^{Ω} of Ω is a σ-field.

Not needed

2. Is the family $\mathcal{F}$ consisting of all finite subsets of Ω and their complements a σ-field always?

3. Let $\mathcal{F}$ be a σ-field on $\Omega = [0,1]$ such that $[\frac{1}{n+1}, \frac{1}{n}] \in \mathcal{F}$ for $n = 1, 2, 3, \ldots$. Show that:

   a) $\{0\} \in \mathcal{F}$.

   b) $\{\frac{1}{n};\ n = 2, 3, 4, \ldots\} \in \mathcal{F}$.

   c) $(\frac{1}{n}, 1] \in \mathcal{F}$ for all $n \geq 2$

   d) $(0, \frac{1}{n}] \in \mathcal{F}$ for all n.

Must know results

4. a) Suppose that $\{\mathcal{A}_n\}_n$ is an increasing sequence of algebras; i.e. $\mathcal{A}_n \subset \mathcal{A}_{n+1}$, for all $n \geq 1$.

(a) Show that $\bigcup_{n=1}^{\infty} \mathcal{A}_n$ is an algebra (field).

b) Find two fields such that their union is <u>NOT</u> a field.

Check answer

Not needed

★ c) Suppose now that $\{\mathcal{A}_n\}_n$ is an increasing sequence of σ-algebras. Show by constructing a counter example that $\bigcup_{n=1}^{\infty} \mathcal{A}_n$ need <u>not</u> be a σ-algebra.

std textbooks

4 (b) $\Omega = [0,1]$

$\mathcal{F}_1 = \{\phi, \Omega \ [0,\tfrac{1}{2}) \ [\tfrac{1}{2}, 1]\}$

$\mathcal{F}_1 = \{\phi, \Omega \ [0, \tfrac{1}{3}) \ [\tfrac{1}{3}, 1]\}$

$\mathcal{F}_1 \cup \mathcal{F}_2 = \{\phi, \Omega, [0,\tfrac{1}{2}), [0,\tfrac{1}{3}), [\tfrac{1}{2},1], [\tfrac{1}{3},1]\}$

Now $[0,\tfrac{1}{2}) \cap [\tfrac{1}{3}, 1] = [\tfrac{1}{3}, \tfrac{1}{2})$ which is not in $\mathcal{F}_1 \cup \mathcal{F}_2$

4 (c) $\Omega = [0,1]$

$A_1 = [0, 1/2]$

$A_2 = [0, 1/3]$

$\vdots$

$A_n = [0, 1/n+1]$

$\vdots$

Define

$\left.\begin{array}{l} \mathcal{C}_1 = \sigma[\{A_1\}] \\ \mathcal{C}_2 = \sigma[\{A_1, A_2\}] \\ \vdots \\ \mathcal{C}_n = \sigma[\{A_1, A_2 \ldots A_n\}] \end{array}\right\}$ are all σ- fields

Since $\{A_1\} \subset \{A_1, A_2\} \subset \ldots \subset \{A_1, A_2 \ldots A_n\} \ldots$

$\sigma[\{A_1\}] \subset \sigma[\{A_1, A_2\}] \subset \ldots$

$A_n \in \bigcup_{i=1}^{\infty} \mathcal{C}_i$ (can be seen)

To show: $\bigcap A_n = \{0\} \notin \bigcup_{i=1}^{\infty} \mathcal{C}_i$

Assume $\{0\} \in \bigcup_{i=1}^{\infty} \mathcal{C}_i \Rightarrow \exists\, n_0$ st $\{0\} \in \mathcal{C}_{n_0}$

Define $B_1 = (1/2, 1]$ $B_2 = (1/3, 1/2]$ $\ldots$ $B_{n_0} = (\frac{1}{n_0+1}, \frac{1}{n_0}]$

Let $B_{n_0+1} = [0, 1/n_0+1]$

$B_1 \ldots B_{n_0}, B_{n_0+1}$ is a partition of $\Omega = [0,1]$

Define $\mathcal{D} = \{B_1, B_2 \ldots B_{n_0}, B_{n_0+1}\}$

We will show $\mathcal{C}_{n_0} = \sigma[\{A_1 \ldots A_{n_0}\}] = \sigma[\mathcal{D}]$

$\mathcal{D} \subseteq \mathcal{C}_{n_0}$ $\left\{\begin{array}{l}\text{since all elements of } \mathcal{D} \text{ can be written as intersections unions} \\ \text{complements of } A_1 \ldots A_{n_0}\end{array}\right.$

$\Rightarrow \sigma[\mathcal{D}] \subseteq \mathcal{C}_{n_0}$

Also $\{A_1, A_2, \ldots A_{n_0+1}\} \subseteq \sigma[\mathcal{D}]$ since each can be obt as unions complements etc of $B_1 \ldots B_{n_0}$ B_{n_0+1}

Know result

• 5. Show that if $\{\mathcal{F}_j\}_{j\in J}$ is any collection of σ-fields defined on the same set Ω, then their intersection $\bigcap_{j\in J} \mathcal{F}_j$ is also a σ-field.

Know results

6. Let $\mathcal{A}$ be a family of subsets of Ω. Recall that

$$\sigma[\mathcal{A}] = \bigcap\{\mathcal{F} : \mathcal{F} \text{ is a } \sigma\text{-field such that } \mathcal{A}\subset\mathcal{F}\}$$

is called the σ-field generated by $\mathcal{A}$.

(*) Let now $\tilde{\mathcal{F}}$ be a σ-field with the following properties:

a) $\mathcal{A} \subset \tilde{\mathcal{F}}$

b) If $\mathcal{F}$ is a σ field, and $\mathcal{A}\subset\mathcal{F}$, then $\tilde{\mathcal{F}} \subset \mathcal{F}$.

Show that: $\sigma[\mathcal{A}] = \tilde{\mathcal{F}}$.

(This explains why $\sigma[\mathcal{A}]$ is refered to as "the smallest σ-field containing $\mathcal{A}$".)

$\therefore \ \sigma[\mathcal{D}] = \mathcal{C}_{n_0}$

$\therefore \{0\} \in \sigma[\mathcal{D}]$ under the assumption we made.

Define $\mathcal{F}$ = set of all finite unions of elements in $\mathcal{D}$

$\mathcal{F} = \{ \bigcup_{j=}^{j_0} B_{n_j} \ ; \ j = 1, 2 \ldots \}$

$\sigma[\mathcal{D}] = \mathcal{F}$ & also show $\mathcal{F}$ is a σ-field.

$\Rightarrow \{0\} \in \mathcal{F}$ (by assumption)

$\Rightarrow$ But $\{0\}$ cannot be expressed as unions or intersection of $\mathcal{F}$.

7. Let $\mathcal{A} = \{[0,1]\} \cup \{[\frac{1}{2^{n+1}}, \frac{1}{2^n}) : n = 0, 1, 2, \ldots\}$.

Which of the following subsets belong to $\sigma[\mathcal{A}]$:

a) $\{0\}$ No b) $\{1\}$ c) $\{\frac{1}{2}\}$

d) $\{\frac{1}{3}\}$ e) $\{0, 1\}$ Yes f) $(\frac{1}{4}, 1]$ ×

g) $[0, \frac{1}{2}]$ × h) $[\frac{1}{4}, 1)$ Yes i) $(0, \frac{1}{2})$ Yes ?

<u>Hint</u>: Let $\tilde{\mathcal{A}} = \{0, 1\} \cup \{[\frac{1}{2^{n+1}}, \frac{1}{2^n}), n = 0, 1, 2 \ldots\}$.

Show first that

check part (ii)

$$\sigma[\mathcal{A}] = \sigma[\tilde{\mathcal{A}}].$$

Shuva

Know results

8. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two collections of subsets of Ω.

a) If $\mathcal{C}_1 \subset \mathcal{C}_2$ show that $\sigma[\mathcal{C}_1] \subset \sigma[\mathcal{C}_2]$.

b) If $\mathcal{C}_1 \subset \sigma[\mathcal{C}_2]$ and $\mathcal{C}_2 \subset \sigma[\mathcal{C}_1]$, then

$\sigma[\mathcal{C}_1] = \sigma[\mathcal{C}_2]$.

8(a) $\mathcal{C}_1 \subset \mathcal{C}_2 \subset \sigma[\mathcal{C}_2] = \bigcap_\alpha \{\mathcal{F}_\alpha : \mathcal{F}_\alpha \text{ is a } \sigma\text{-field} \ni \mathcal{C}_2 \subset \mathcal{F}_\alpha\}$

Done!

9. If C is the Cantor set and λ is the Lebesgue measure (introduced in Lecture 3'), show that $\lambda(C) = 0$.

1). Given 2 fields

~~in~~ then $\cap$ a σ-field ?

$\cup$ a σ-field ?

You can simply state results from HW & use them

2) ** Prove the funct^n is measurable

$X : (\Omega, \mathcal{A}) \longrightarrow (\mathbb{R}, \mathcal{B})$

$X^{-1}(\mathcal{B}) \subset \mathcal{A}$ then X is $\mathcal{B}$-$\mathcal{A}$ meas.

wrt $\mathcal{F}(X) = X^{-1}(\mathcal{B})$ which is a σ-field

ie $\mathcal{A}$ has to be $X^{-1}(\mathcal{B})$ or larger to be meas.

1] Let $\mathcal{A} = 2^{\Omega}$

(i) $\Omega \in \mathcal{A}$ trivially

(ii) $A \in \mathcal{A} \Rightarrow A \subset \Omega$

$\Rightarrow A^c \subset \Omega$

$\Rightarrow A^c \in \mathcal{A}$ $\therefore$ $\mathcal{A}$ is closed under complement

(iii) $A_1, A_2 \ldots \in \mathcal{A} \rightarrow A_1, A_2 \ldots \subseteq \Omega$

$\Rightarrow \bigcup_{i=1}^{\infty} A_i \subseteq \Omega$

$\Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$ $\therefore$ $\mathcal{A}$ is closed under countable union

2] A set whose compliment is finite is a cofinite set

$\mathcal{F}$ = collection of all finite and cofinite subsets of Ω

(i) Trivially $\Omega \subseteq \Omega$ and $\Omega^c = \phi \subset \Omega$

$\therefore$ Ω is a cofinite set $\Rightarrow \Omega \in \mathcal{F}$

(ii) $A \in \mathcal{F} \rightarrow$ either A is finite or cofinite and $A \subset \Omega$

If A is finite then A^c is cofinite ($\because (A^c)^c = A$).

Also $A^c \subset \Omega$

$\therefore$ $A^c \in \mathcal{F}$

If A is cofinite then A^c is finite (by def)

Also $A^c \subset \Omega$

$\therefore$ $A^c \in \mathcal{F}$

(iii) $A_1, A_2 \ldots \in \mathcal{F} \rightarrow A_i \subset \Omega$ & $\bigcup_{i=1}^{\infty} A_i \subset \Omega$

If Ω is finite then all A_i are finite and $\bigcup_{i=1}^{\infty} A_i$ is also finite

$\therefore$ $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

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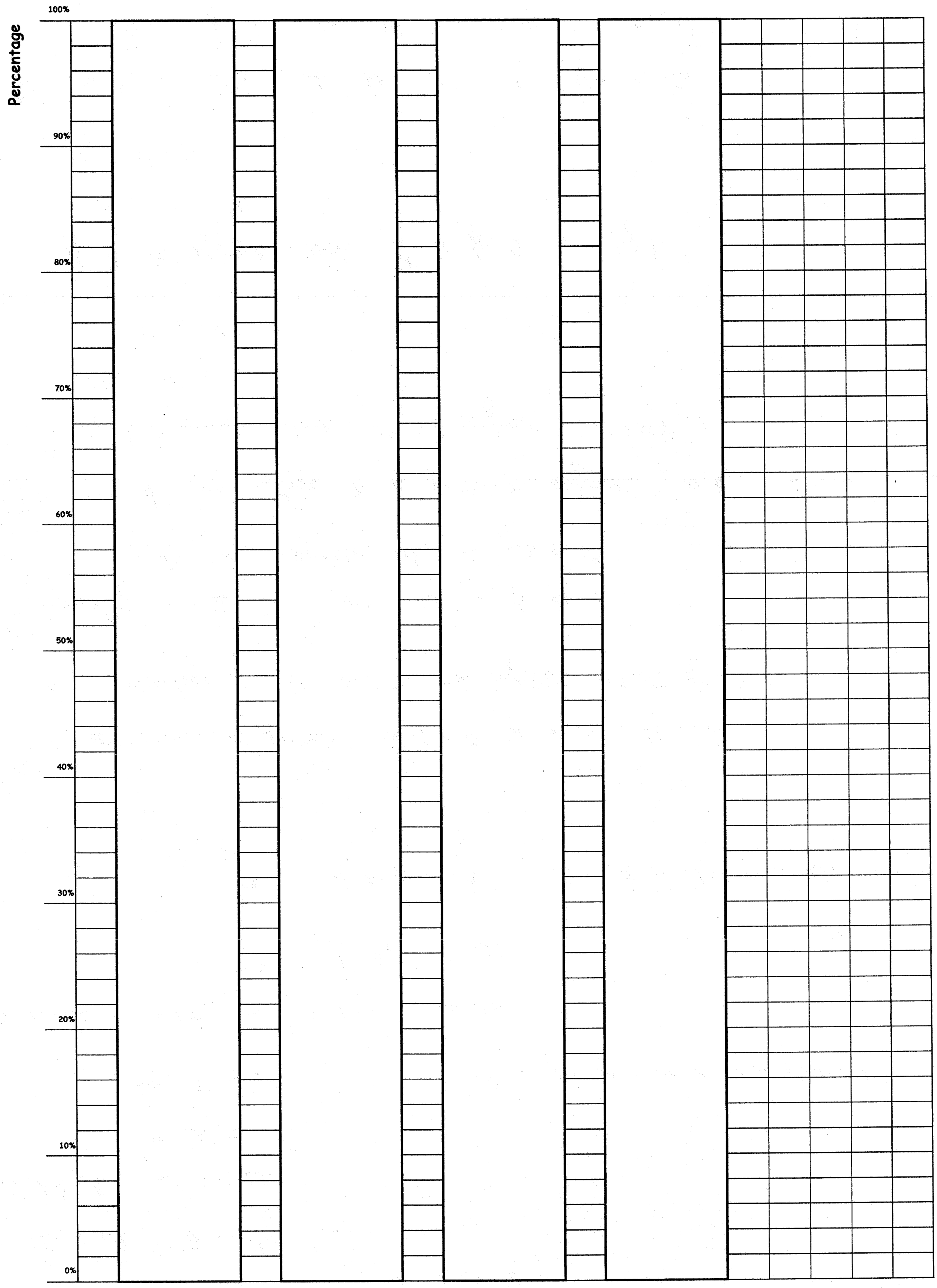

If Ω is not finite. Suppose $\Omega = \mathbb{N}$

Let $A_i = \{2i\}$ (singleton). Clearly $A_i \in \mathcal{F}$ since it is finite

But $\bigcup_{i=1}^{\infty} A_i = \{2i \,;\, i = 1, 2 \dots\}$ ie set of all even #'s is infinite

and $\left(\bigcup_{i=1}^{\infty} A_i\right)^c = \{2i+1 \,;\, i = 0, 1, \dots\}$ ie set of all odd #'s is also infinite

$\therefore \left(\bigcup_{i=1}^{\infty} A_i\right)$ is neither finite nor cofinite and $\notin \mathcal{F}$

$\therefore$ $\mathcal{F}$ = set of all finite and cofinite subsets of Ω is a σ-field only when Ω is finite.

3] $\mathcal{F}$ is a σ-field on $\Omega = [0,1]$ and $\left[\frac{1}{n+1}, \frac{1}{n}\right] \in \mathcal{F} \quad \forall\, n \geq 1$.

Since $\mathcal{F}$ is a σ-field $\quad \Omega \in \mathcal{F}$

$$\text{If } A \in \mathcal{F} \rightarrow A^c \in \mathcal{F}$$

$$A_1, A_2 \dots \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$$

(a) $\left[\frac{1}{n+1}, \frac{1}{n}\right] \in \mathcal{F} \quad \forall\, n \geq 1 \longrightarrow \bigcup_{n=1}^{\infty} \left[\frac{1}{n+1}, \frac{1}{n}\right] = (0, 1] \in \mathcal{F}$

$\therefore [0,1] \setminus (0\ 1] = \{0\} \in \mathcal{F}$

(b) $\left[\frac{1}{n+1}, \frac{1}{n}\right] \in \mathcal{F} \quad \forall\, n \geq 1 \Rightarrow \left[\frac{1}{n+1}, \frac{1}{n}\right] \setminus \left[\frac{1}{n}, \frac{1}{n-1}\right]^c = \{1/n\} \in \mathcal{F} \quad \forall\, n \geq 2$

$$\Rightarrow \bigcup_{n=2}^{\infty} \{1/n\} \in \mathcal{F}$$

ie $\{1/n \,;\, n = 2, 3 \dots\} \in \mathcal{F}$

(c) $\left[\frac{1}{2}, 1\right], \left[\frac{1}{3}, \frac{1}{2}\right], \dots \left[\frac{1}{n-1}, \frac{1}{n}\right] \in \mathcal{F} \Rightarrow \bigcup_{k=1}^{n-1} \left[\frac{1}{k+1}, \frac{1}{k}\right] \in \mathcal{F}$

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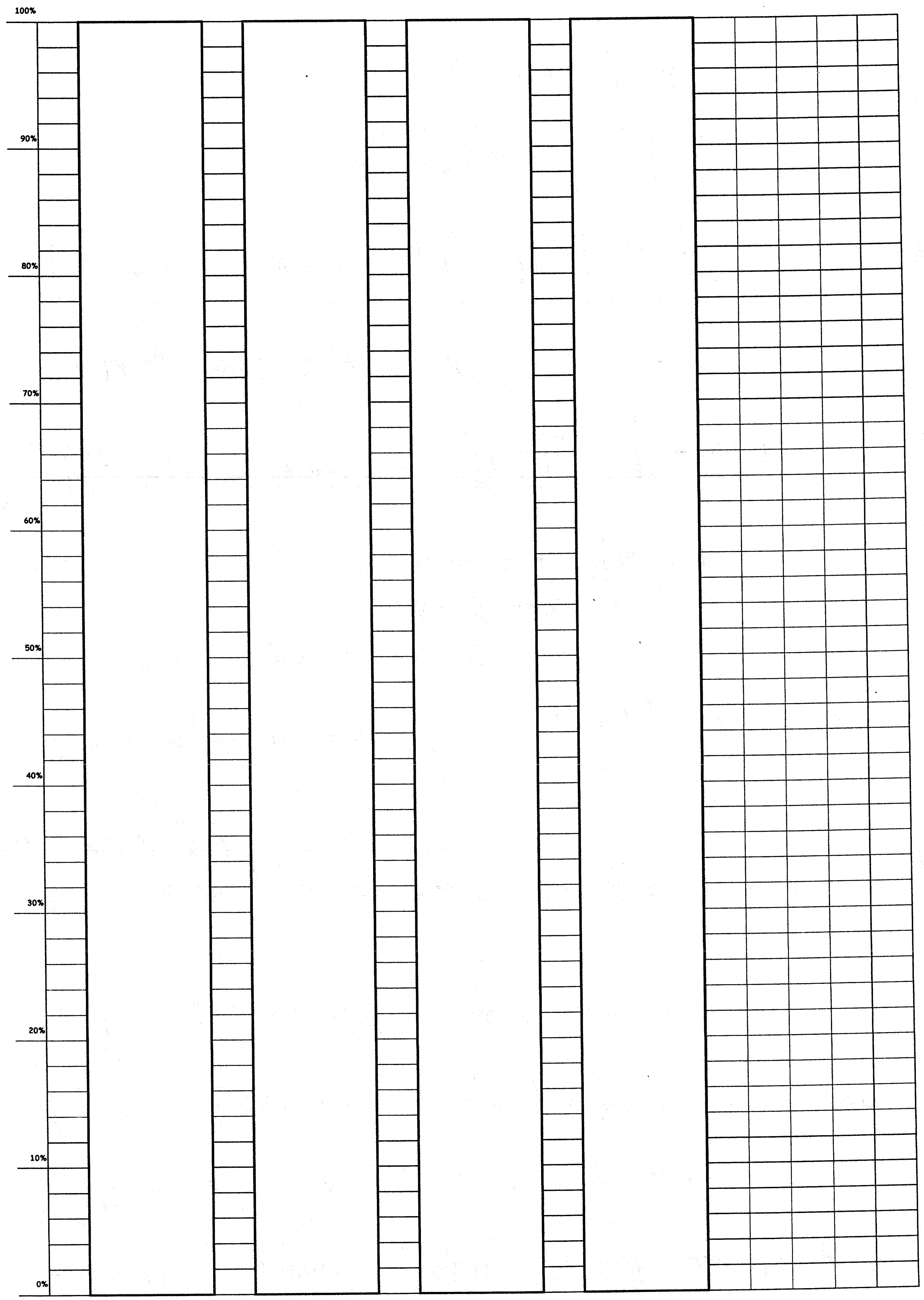

ie $\left[\frac{1}{n}, 1\right] \in \mathcal{F}$

We also showed $\{1/n\} \in \mathcal{F} \quad \forall\, n \geq 2$

$\therefore \quad \left[\frac{1}{n}, 1\right] \setminus \{1/n\} = \left(\frac{1}{n}, 1\right] \in \mathcal{F}$

(d) We have shown $(0,1] \in \mathcal{F}$ and $\left(\frac{1}{n}, 1\right] \in \mathcal{F}$

$\therefore \quad (0,1] \setminus \left(\frac{1}{n}, 1\right] \in \mathcal{F}$

#4] $\{\mathcal{A}_n\}$ is an $\uparrow$ seq of fields

ie $\mathcal{A}_n \subset \mathcal{A}_{n+1} \quad \forall\, n \geq 1$ and $\mathcal{A}_n$ is a field [closed under compl & finite unions

let $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{A}_n$

(a) $\Omega \in \mathcal{A}_n \quad \forall\, n \geq 1$ (since $\mathcal{A}_n$'s are fields)

$\therefore \Omega \in \left(\bigcup_{n=1}^{\infty} \mathcal{A}_n\right)$

If $A \in \mathcal{B} \Rightarrow \exists$ some k_0 st $A \in \mathcal{A}_{k_0}$

$\Longrightarrow A^c \in \mathcal{A}_{k_0}$ (since $\mathcal{A}_{k_0}$ is a field)

$\Longrightarrow A^c \in \left(\bigcup_{n=1}^{\infty} \mathcal{A}_n\right) = \mathcal{B}$

If $A_1, A_2, A_3 \dots A_n \in \mathcal{B} \longrightarrow \exists$ some $k_1, k_2 \dots k_n$ st

$A_1 \in \mathcal{A}_{k_1}$
$A_2 \in \mathcal{A}_{k_2}$
$\vdots$
$A_n \in \mathcal{A}_{k_n}$

let $m = \max\{k_1, k_2 \dots k_n\}$

Since $\{\mathcal{A}_n\}_n$ is an $\uparrow$ seq $\Rightarrow$

$\mathcal{A}_{k_1} \subset \mathcal{A}_m$
$\mathcal{A}_{k_2} \subset \mathcal{A}_m$
$\vdots$
$\mathcal{A}_{k_n} \subset \mathcal{A}_m$

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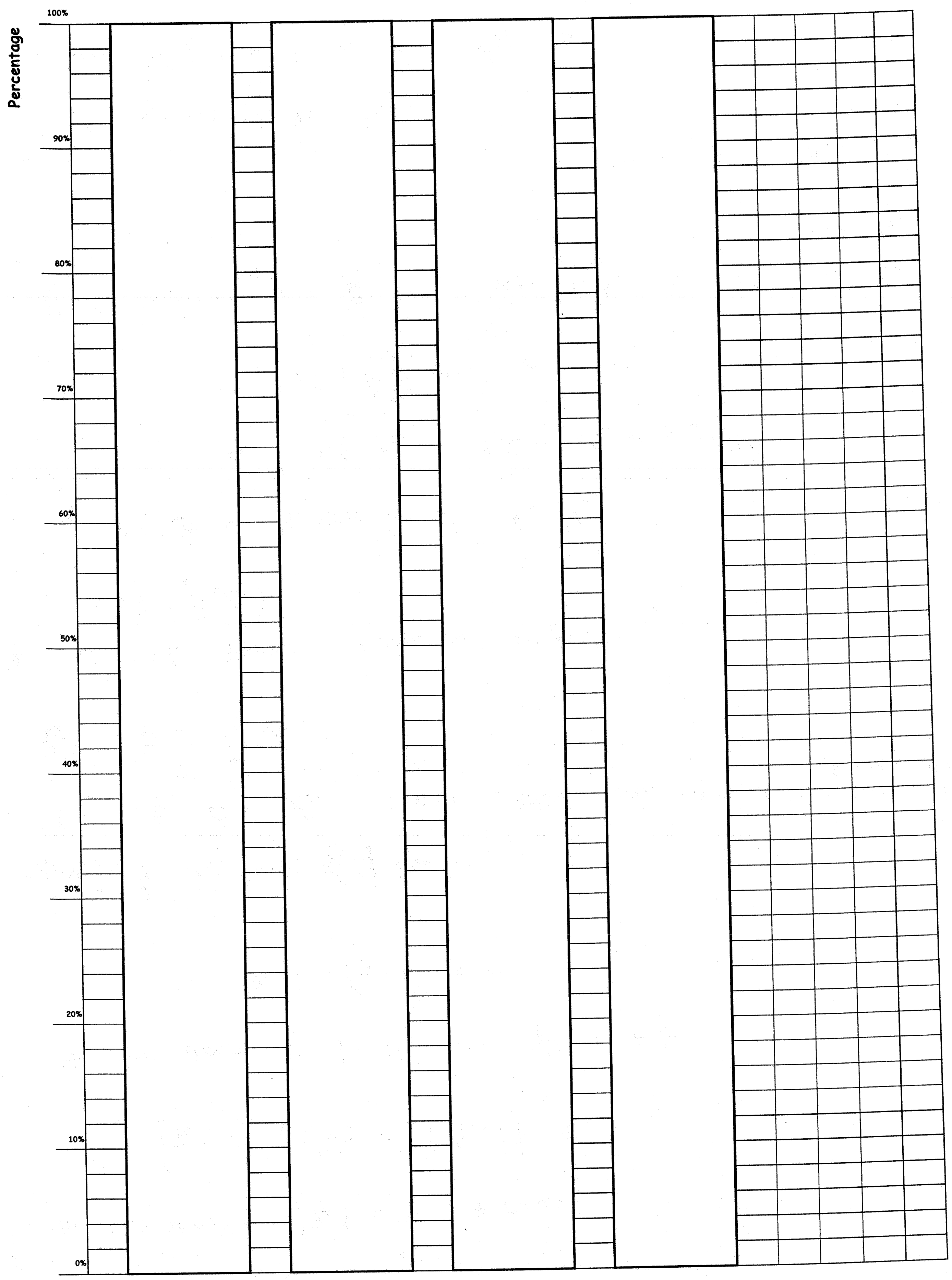

$$\Rightarrow \quad A_1 \in \mathcal{A}_m \;,\; A_2 \in \mathcal{A}_m \;\ldots\; A_n \in \mathcal{A}_m$$

$$\rightarrow \quad \bigcup_{i=1}^{n} A_i \in \mathcal{A}_m$$

$$\rightarrow \quad \bigcup_{i=1}^{n} A_i \in \mathcal{B} = \bigcup_{l=1}^{\infty} \mathcal{A}_l$$

$\therefore \quad \bigcup_{n=1}^{\infty} \mathcal{A}_n$ is a field

4(b) $\quad \Omega = \{0,1,2,3\}$

$\mathcal{F}_1 = \{ \phi, \{1\}, \{0\}, \{1,2,3\}, \{0,2,3\}, \{0,1\}, \Omega \}$

$\mathcal{F}_2 = \{ \phi, \{2\}, \{3\}, \{0,1,3\}, \{0,1,2\}, \{2,3\}, \Omega \}$

$\mathcal{F}_1$ and $\mathcal{F}_2$ are closed under complement and finite union $\Rightarrow$ fields

$\mathcal{F}_1 \cup \mathcal{F}_2 = \{ \phi, \{0\}, \{1\}, \{2\}, \{3\}, \{0,1\}, \{2,3\}, \{1,2,3\} \{0,2,3\}, \{0,1,2\}, \{0,1,3\}, \Omega \}$

$\mathcal{F}_1 \cup \mathcal{F}_2$ does not contain $\{1,2\} = \{1\} \cup \{2\} \Rightarrow$ not closed under union

$\therefore \mathcal{F}_1 \cup \mathcal{F}_2$ is not a field.

$\therefore$ Unions of arbitrary fields is NOT necessarily a field !! (this is true for σ-fields also

4(c)] $\quad \{\mathcal{A}_n\}_n$ is an $\uparrow$ seq of σ-fields but $\bigcup_{n=1}^{\infty} \mathcal{A}_n$ is not always a σ-field

Consider $\Omega = \{1,2,3,4,5\ldots\}$ = set of positive integers

let $\mathcal{B}_i$ = σ-field generated by $\{\{1\},\{2\}\ldots\{i\}\}$

ie $\mathcal{B}_1 = \sigma[\{\{1\}\}] = \{\phi, \{1\}, \{1\}^c, \Omega\}$

$\mathcal{B}_2 = \sigma[\{\{1\},\{2\}\}] = \{\phi, \{1\}, \{2\}, \{1\}^c, \{2\}^c, \{1,2\}, \{1,2\}^c, \Omega\}$

ie $\mathcal{B}_i = \{ A \subseteq \Omega : A \subset \{1,2\ldots i\}$ OR $A^c \subset \{1,2\ldots i\} \}$

It is easy to see $\mathcal{B}_i \subset \mathcal{B}_{i+1} \quad \forall i \quad \Rightarrow \{\mathcal{B}_i\}_i$ is $\uparrow$ seq of σ-fields

We have to show $\mathcal{B} = \bigcup_{i=1}^{\infty} \mathcal{B}_i$ is not a σ-field (by part (a) it is a field)

ie to show $\mathcal{B}$ is not closed under countable unions.

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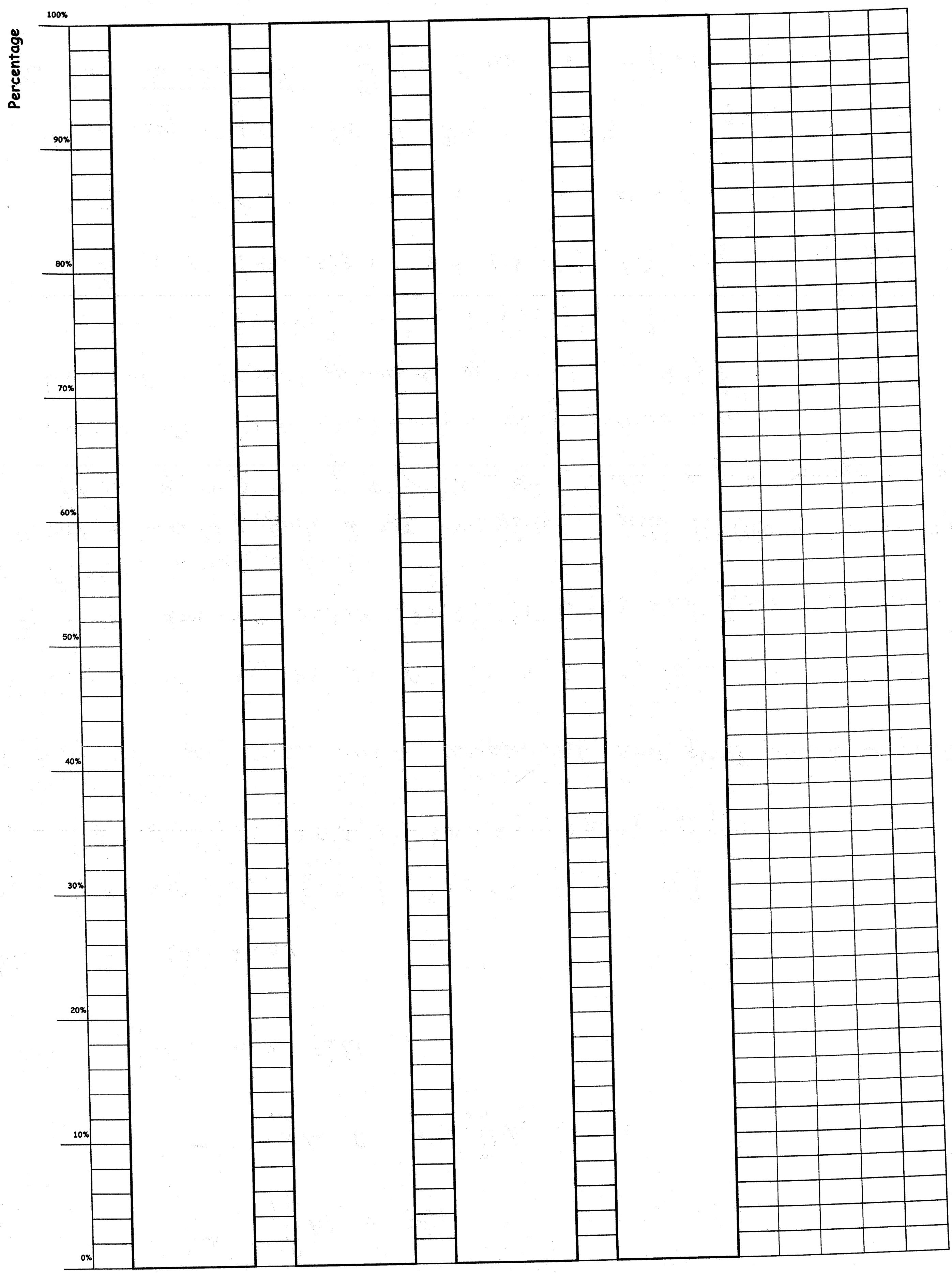

let $A_k = \{2k\}$ and $A = \bigcup_{k=1}^{\infty} A_k = \{2, 4, 6 \ldots\}$

$A_k = \{2k\} \subset \{1, 2, \ldots 2k\} \subset \mathcal{B}_{2k} \subset \mathcal{B}$

ie $A_k \in \mathcal{B} \quad \forall k \geq 1$

Now let if possible $A = \bigcup_{k=1}^{\infty} A_k \in \mathcal{B} = \bigcup_{i=1}^{\infty} \mathcal{B}_i$

$\Rightarrow \exists$ some $n_0 \geq 1$ st $A \in \mathcal{B}_{n_0}$

$\Rightarrow A \subset \{1, 2, \ldots n_0\}$ OR $A^c \subset \{1, 2 \ldots n_0\}$

But $A = \{2, 4, 6, 8 \ldots\}$ & $A^c = \{1, 3, 5 \ldots\}$

$\therefore \quad A = \bigcup_{k=1}^{\infty} A_k \notin \mathcal{B}$

ie $\mathcal{B}$ is not closed under countable unions

$\therefore \bigcup_{i=1}^{\infty} \mathcal{B}_i$ where $\{\mathcal{B}_i\}_i$ is $\uparrow$ seq of σ-fields is not a σ-field.

5] $\{\mathcal{F}_j\}_{j \in J}$ is a collection of σ-fields defined on Ω

To show $\bigcap_{j \in J} \mathcal{F}_j$ is also a σ-field

(i) Each $\mathcal{F}_j$ is a σ-field $\Rightarrow \Omega \in \mathcal{F}_j \quad \forall j \in J$

$\Rightarrow \Omega \in \bigcap_{j \in J} \mathcal{F}_j$

(ii) $A \in \bigcap_{j \in J} \mathcal{F}_j \Rightarrow A \in \mathcal{F}_j \quad \forall j \in J$

$\Rightarrow A^c \in \mathcal{F}_j \quad \forall j \in J$

$\Rightarrow A^c \in \bigcap_{j \in J} \mathcal{F}_j$

(iii) $A_1, A_2 \ldots \in \bigcap_{j \in J} \mathcal{F}_j \rightarrow A_1 A_2 \ldots \in \mathcal{F}_j \quad \forall j \in J$

$\Rightarrow \bigcup_{k=1}^{\infty} A_k \in \mathcal{F}_j \quad \forall j \in J$

$\Rightarrow \bigcup_{k=1}^{\infty} A_k \in \bigcap_{j \in J} \mathcal{F}_j$

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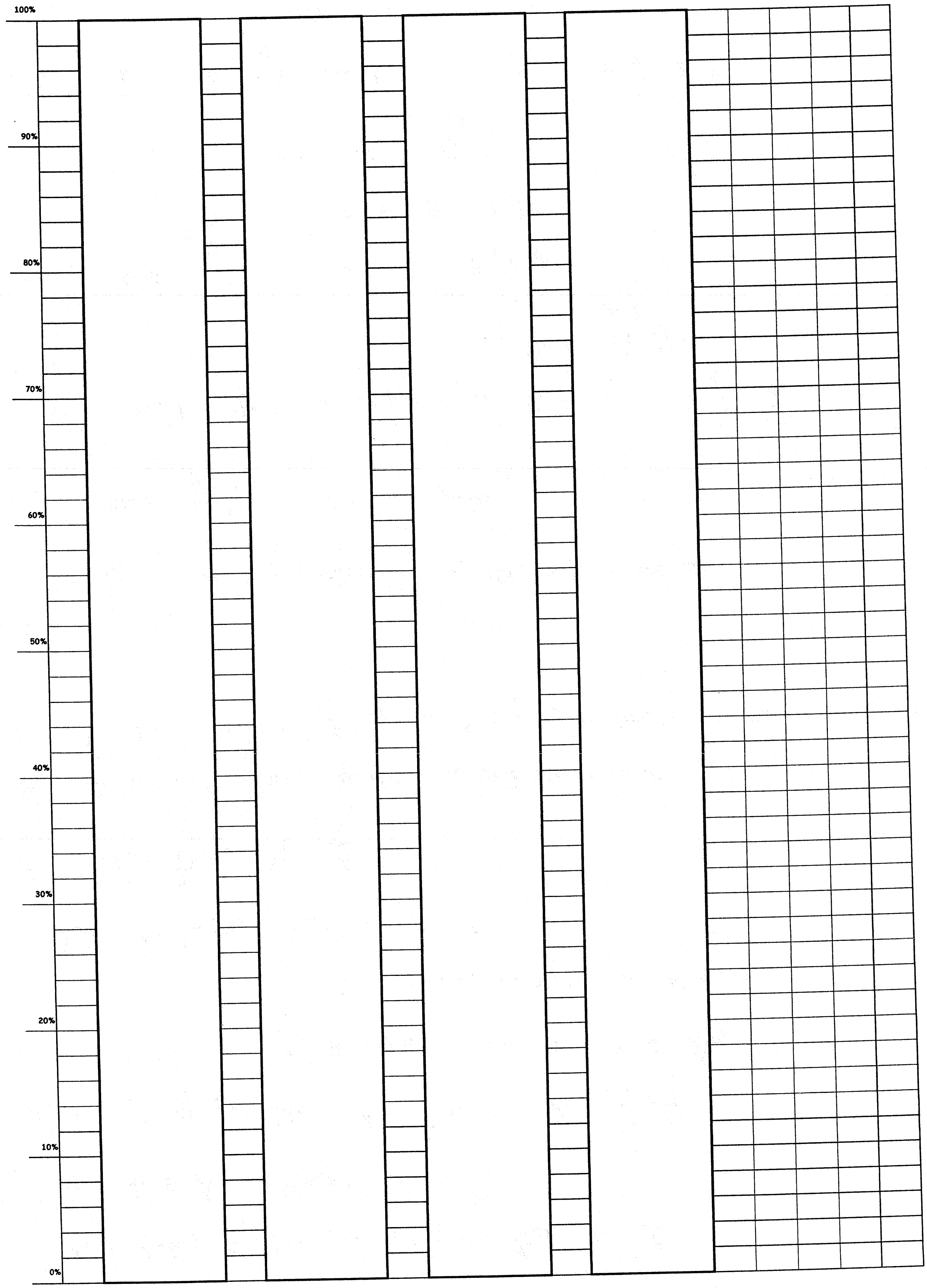

6] $\mathcal{A}$ = collection of subsets of Ω

$\sigma[\mathcal{A}] = \bigcap \{ \mathcal{F} : \mathcal{F} \text{ is a } \sigma\text{-field and } \mathcal{A} \subseteq \mathcal{F} \}$

$\tilde{\mathcal{F}}$ is a σ-field st (a) $\mathcal{A} \subseteq \tilde{\mathcal{F}}$

(b) If $\mathcal{F}$ is a σ-field and $\mathcal{A} \subset \mathcal{F}$ then $\tilde{\mathcal{F}} \subset \mathcal{F}$

Show $\sigma[\mathcal{A}] = \tilde{\mathcal{F}}$

We will show $\tilde{\mathcal{F}} \subseteq \sigma[\mathcal{A}]$ and $\sigma[\mathcal{A}] \subseteq \tilde{\mathcal{F}}$

(i) By def: $\mathcal{A} \subseteq \sigma[\mathcal{A}]$ and $\mathcal{A} \subseteq \tilde{\mathcal{F}}$

$\sigma[\mathcal{A}]$ is a σ-field containing $\mathcal{A}$ $\Rightarrow$ $\tilde{\mathcal{F}} \subseteq \sigma[\mathcal{A}]$ } by (b)

(ii) $\tilde{\mathcal{F}}$ is a σ-field containing $\mathcal{A}$

$\Rightarrow$ $\tilde{\mathcal{F}}$ is one of the elements of $\{ \mathcal{F} : \mathcal{F} \text{ is a } \sigma\text{-field } \& \ \mathcal{A} \subseteq \mathcal{F} \}$

Now since $\sigma[\mathcal{A}] = \bigcap \{ \mathcal{F} : \mathcal{F} \text{ is a } \sigma\text{-field } \& \ \mathcal{A} \subset \mathcal{F} \}$

$\Rightarrow$ $\sigma[\mathcal{A}] \subseteq \tilde{\mathcal{F}}$ [If $A = \bigcap B_k \Rightarrow A \subseteq B_k \ \forall k$]

$\therefore$ $\sigma[\mathcal{A}] = \tilde{\mathcal{F}}$

7] $\mathcal{A} = \{[0,1]\} \cup \{ [\frac{1}{2^{n+1}}, \frac{1}{2^n}) : n = 0, 1, 2 \ldots \}$

$\tilde{\mathcal{A}} = \{0, 1\} \cup \{ [\frac{1}{2^{n+1}}, \frac{1}{2^n}) : n = 0, 1, 2 \ldots \}$

We will show $\sigma[\mathcal{A}] = \sigma[\tilde{\mathcal{A}}]$

We can then use the fact that: if we have a set $\mathcal{C}$ which consists of countable disjoint partitions of Ω then the $\sigma[\mathcal{C}]$ consists only of countable unions of elements of $\mathcal{C}$.

$\therefore$ if we can express a set as a countable union of elem in $\tilde{\mathcal{A}}$ then the set belongs to $\sigma[\tilde{\mathcal{A}}] = \sigma[\mathcal{A}]$

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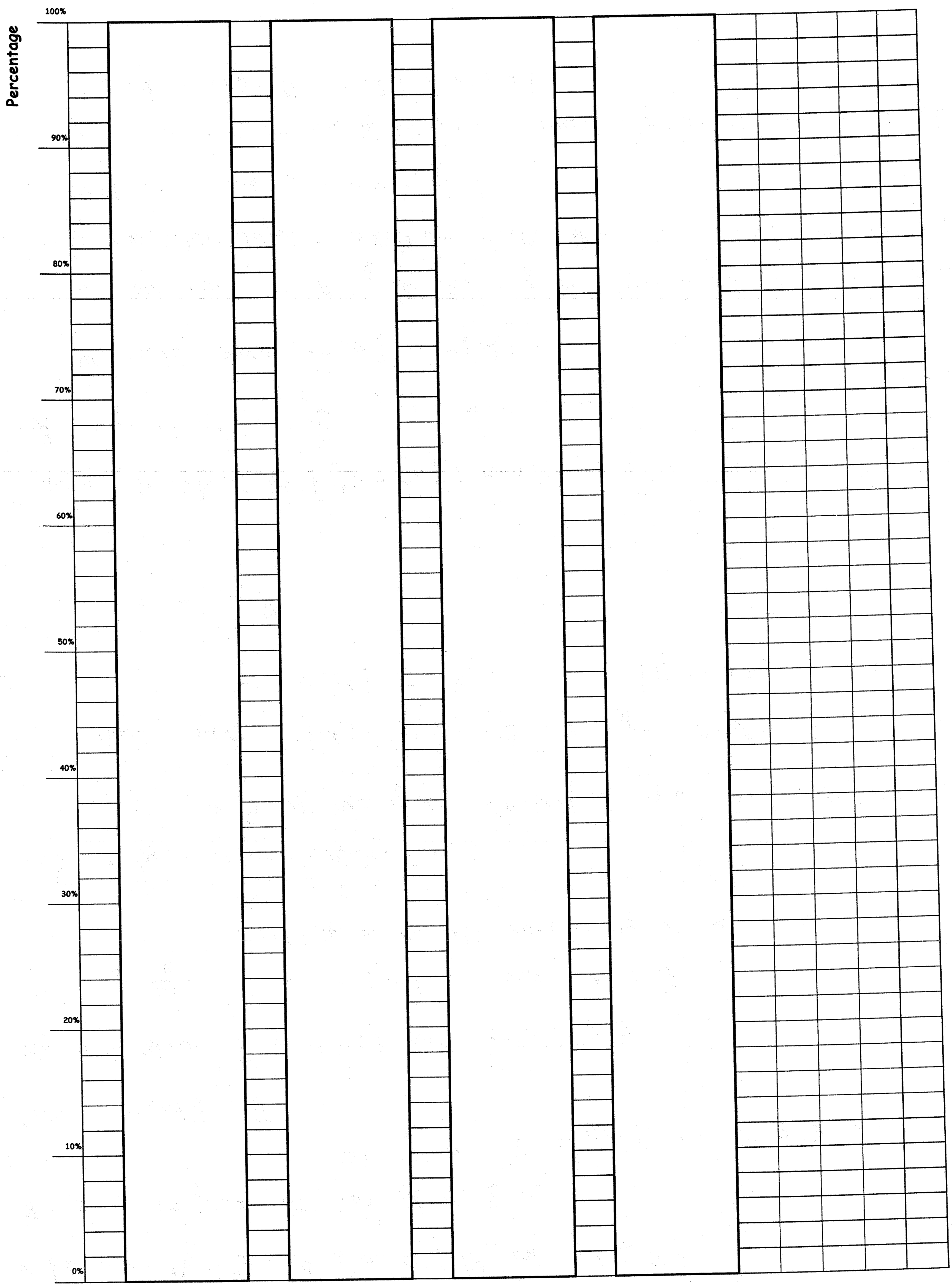


(i) We know $[0,1] \in \sigma[\mathcal{A}]$ and $\left[\frac{1}{2^{n+1}}, \frac{1}{2^n}\right) \in \sigma[\mathcal{A}]$

$$\therefore \bigcup_{n=0}^{\infty} \left[\frac{1}{2^{n+1}}, \frac{1}{2^n}\right) = (0,1) \in \sigma[\mathcal{A}] \quad (\because \text{closed under countable unions})$$

$$\therefore (0,1)^c \cap [0,1] = \{0,1\} \in \sigma[\mathcal{A}]$$

$$\therefore \tilde{\mathcal{A}} \subset \sigma[\mathcal{A}] \text{ ——— } (*)$$

Now we know $\{0,1\} \in \sigma[\tilde{\mathcal{A}}]$ and $\left[\frac{1}{2^{n+1}}, \frac{1}{2^n}\right) \in \sigma[\tilde{\mathcal{A}}]$

$$\therefore \bigcup_{n=0}^{\infty} \left[\frac{1}{2^{n+1}}, \frac{1}{2^n}\right) = (0,1) \in \sigma[\tilde{\mathcal{A}}]$$

$$\therefore \{0,1\} \cup (0,1) = [0,1] \in \sigma[\tilde{\mathcal{A}}]$$

$$\therefore \mathcal{A} \subset \sigma[\tilde{\mathcal{A}}] \text{ ——— } (**)$$

$\therefore$ by $(*)$ & $(**)$ $\Rightarrow$ $\sigma[\mathcal{A}] = \sigma[\tilde{\mathcal{A}}]$ (see #8 (b) proof)

(ii) $\mathcal{C}$ consists of countable # of disjoint partitions of Ω

$\sigma[\mathcal{C}] = \{$ countable union of elements in $\mathcal{C}\}$

let $\mathcal{D} = \{$countable union of elements in $\mathcal{C}\}$

$\mathcal{C} \in \sigma[\mathcal{C}]$ and since $\sigma[\mathcal{C}]$ is closed under countable unions

$\mathcal{D} \subset \sigma[\mathcal{C}]$

Now $\mathcal{C} \in \sigma[\mathcal{C}]$

If $A \in \sigma[\mathcal{C}] \Rightarrow A^c \in \sigma[\mathcal{C}]$

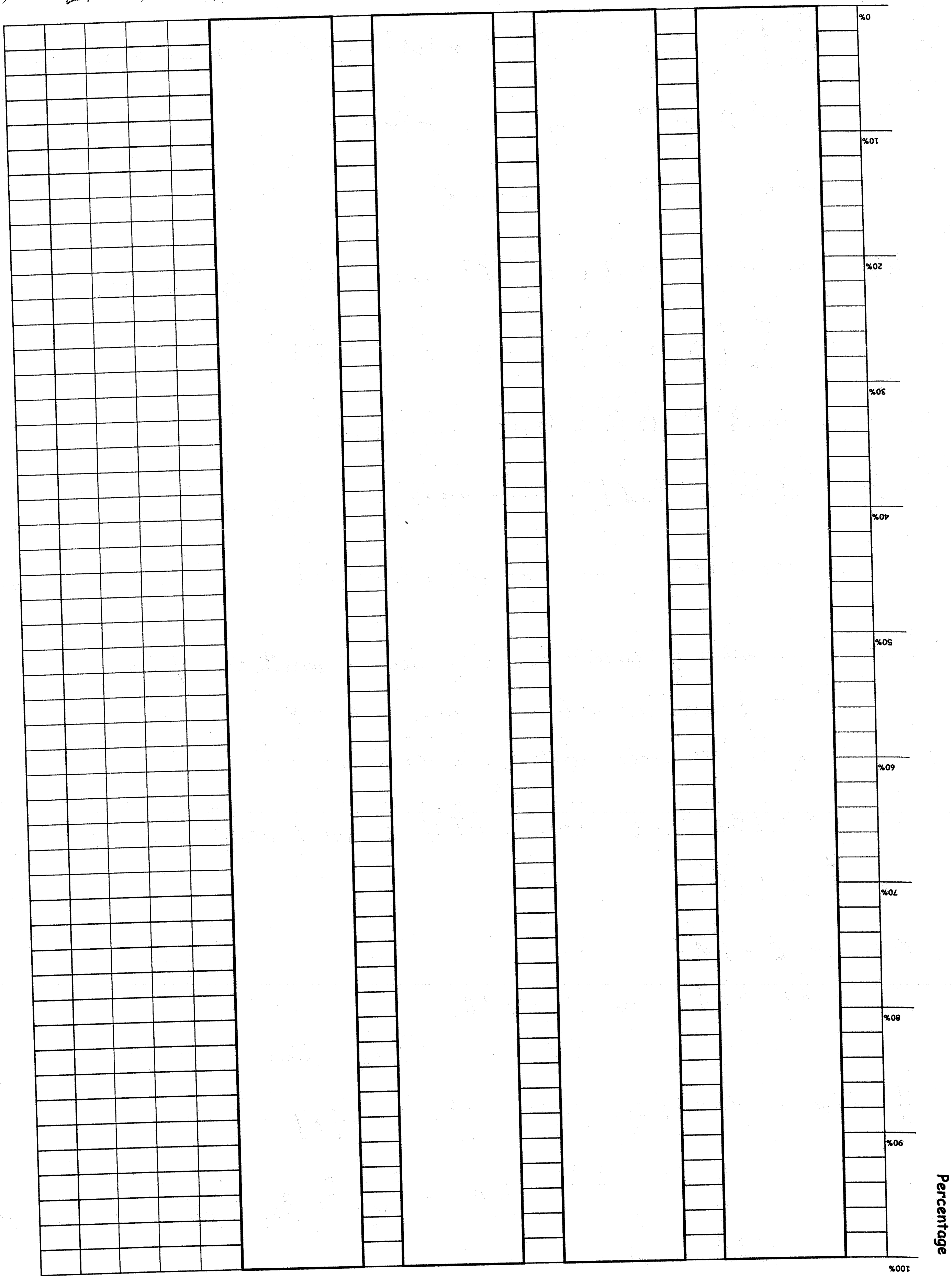
But $A^c = \bigcup_{i=1}^{\infty} B_i$ where $B_i \in \mathcal{C}$

If $A_1, A_2, \ldots \in \sigma[\mathcal{C}]$ then $\bigcup_{i=1}^{\infty} A_i \in \sigma[\mathcal{C}]$

But $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} \bigcup_{k=1}^{\infty} B_{ki}$ which is countable

$\therefore \sigma[\mathcal{C}] \subset \mathcal{D}$

$\left[\frac{1}{2}, 1\right)$ $\left[\frac{1}{4}, \frac{1}{2}\right)$ $\left[\frac{1}{8}, \frac{1}{4}\right)$



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(a) $\{0\}$ cannot be expressed as a countable union of intv in $\tilde{\mathcal{A}}$

$\{0\} \notin \sigma[\tilde{\mathcal{A}}]$

(b) $\{1\} \notin \sigma[\tilde{\mathcal{A}}]$

(c) $\{1/2\} \notin \sigma[\tilde{\mathcal{A}}]$

(d) $\{1/3\} \notin \sigma[\tilde{\mathcal{A}}]$

(e) $\{0, 1\} \subset \tilde{\mathcal{A}} \Rightarrow \{0,1\} \in \sigma[\tilde{\mathcal{A}}]$

(f) $(\frac{1}{4}, 1] = [0,1] \setminus [0, 1/4)$

But $[0, 1/4)$ cannot be expressed as a countable union of disjoint sets

$\therefore (1/4, 1] \notin \sigma[\tilde{\mathcal{A}}]$

(g) $[0, 1/2]$ cannot be expressed as a countable union of disjoint sets in $\tilde{\mathcal{A}}$

$\therefore [0, 1/2] \notin \sigma[\tilde{\mathcal{A}}]$

(h) $[\frac{1}{4}, 1) = [1/4, 1/2) \cup [1/2, 1) \quad \therefore \in \sigma[\tilde{\mathcal{A}}]$

(i) $(0, 1/2) = (0,1) \setminus [1/2, 1) \quad \in \sigma[\tilde{\mathcal{A}}]$

#8] $\mathcal{C}_1$ and $\mathcal{C}_2$ are 2 collections of subsets of Ω

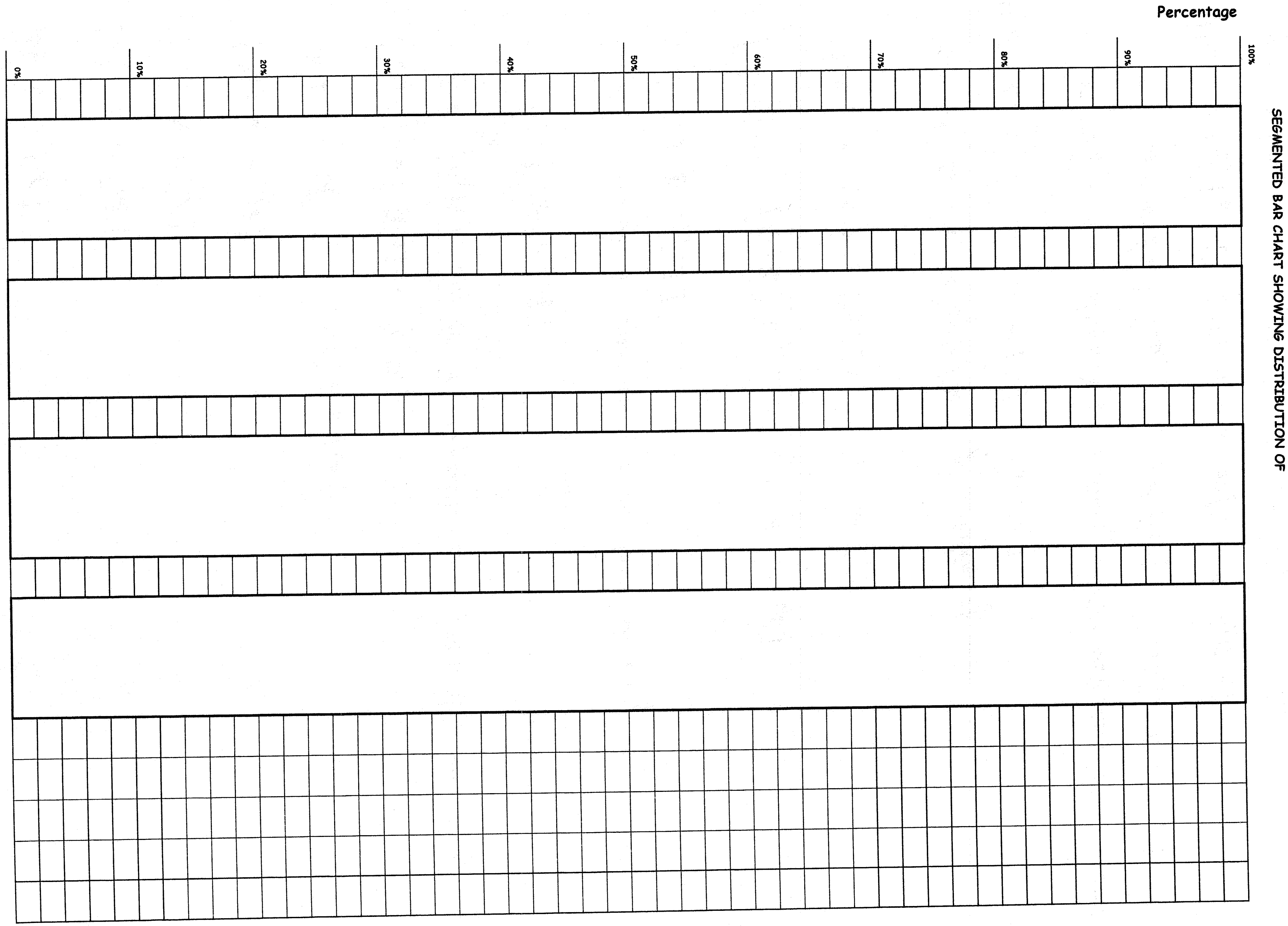
(a) If $\mathcal{C}_1 \subset \mathcal{C}_2$ show $\sigma[\mathcal{C}_1] \subset \sigma[\mathcal{C}_2]$

$\sigma[\mathcal{C}_2] = \{$ all elements of $\mathcal{C}_2$ thier complements, countable unions etc $\}$

ie $\mathcal{C}_2 \subset \sigma[\mathcal{C}_2] \Rightarrow \mathcal{C}_1 \subset \sigma[\mathcal{C}_2]$ $\{$since $\mathcal{C}_1 \subset \mathcal{C}_2\}$

Also $\sigma[\mathcal{C}_1] = \bigcap \{\mathcal{F} : \mathcal{F}$ is a σ-field & $\mathcal{C}_1 \subset \mathcal{F}\}$

Now $\sigma[\mathcal{C}_2]$ is a σ-field and $\mathcal{C}_1 \subset \sigma[\mathcal{C}_2] \Rightarrow \sigma[\mathcal{C}_2]$ is a candidate in the above intersection.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$\therefore \quad \sigma[\mathcal{C}_1] \subset \sigma[\mathcal{C}_2] \qquad \{\because \text{ If } A = \bigcap_\alpha B_\alpha \Rightarrow A \subseteq B_\alpha \ \forall \alpha\}$

8(6) If $\mathcal{C}_1 \subset \sigma[\mathcal{C}_2]$ and $\mathcal{C}_2 \subset \sigma[\mathcal{C}_1] \Rightarrow \sigma[\mathcal{C}_1] = \sigma[\mathcal{C}_2]$

By definition $\sigma[\mathcal{A}] = \bigcap\{\mathcal{F} : \mathcal{F} \text{ is a } \sigma\text{-field and } \mathcal{A} \subset \mathcal{F}\}$

$\sigma[\mathcal{C}_2]$ is a σ-field & $\mathcal{C}_1 \subset \sigma[\mathcal{C}_2] \Rightarrow \sigma[\mathcal{C}_1] \subset \sigma[\mathcal{C}_2]$

$\sigma[\mathcal{C}_1]$ is a σ-field & $\mathcal{C}_2 \subset \sigma[\mathcal{C}_1] \Rightarrow \sigma[\mathcal{C}_2] \subset \sigma[\mathcal{C}_1]$

$\therefore \quad \sigma[\mathcal{C}_1] = \sigma[\mathcal{C}_2]$

9] To show that the cantor set has measure zero.

$C_0 = [0, 1]$

Let C_{n+1} be obt recursively from C_n by removing the midde thirds of the intervals which made up C_n

ie $C_1 = [0, 1/3] \cup [2/3, 1]$

$C_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$

$\vdots$

The cantor set is $\mathcal{C} = \bigcap_{n=0}^{\infty} C_n$ {consists of ∞ly many pts in $[0,1]$}.

$$\therefore \quad C^c = \bigcup_{n=0}^{\infty} C_n^c$$

$$= \bigcup_{n=0}^{\infty} K_n \quad \{\text{say } K_n = C_n^c\}$$

Now $K_0 = \phi$

$K_1 = (1/3, 2/3)$

$K_2 = (1/9, 2/9) \cup (1/3, 2/3) \cup (7/9, 8/9)$

$\vdots$

$\therefore \{K_n\}_n$ is an $\uparrow$ sequence $\Rightarrow \bigcup_{n=0}^{\infty} K_n = \lim_{N\to\infty} \bigcup_{n=0}^{N} K_n = \lim_{N\to\infty} K_N$

$$\therefore \lambda(C^c) = \lim_{N\to\infty} \lambda(K_N) = \frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \cdots = \frac{1}{3}\sum_{n=0}^{\infty}(2/3)^n = 1$$

$$\therefore \quad \lambda(C) = 1 - \lambda(C^c) = 0$$

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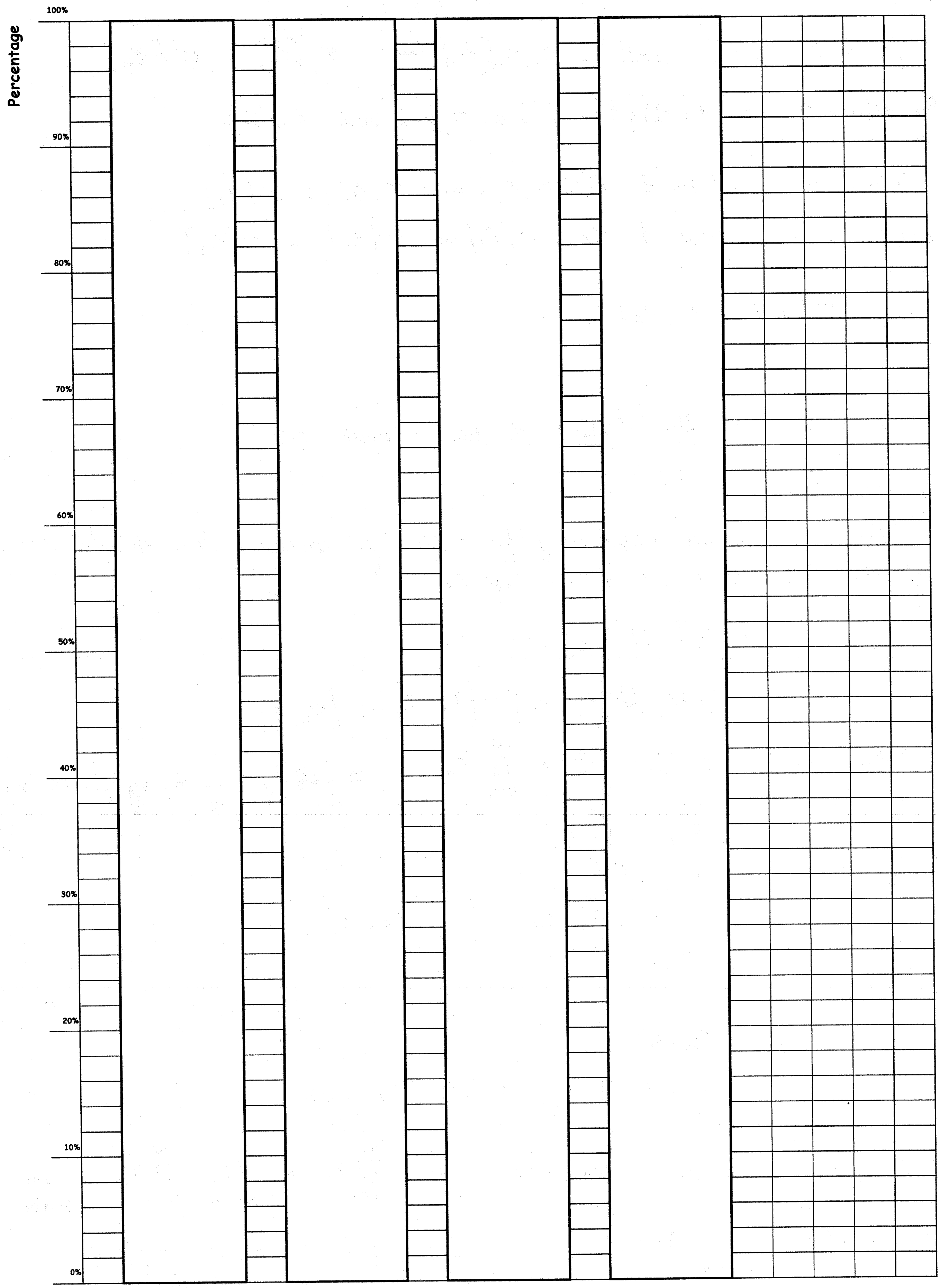

STA 5446, Fall 2005.

Homework 1, Problem 4 a) and c).

1. (a) Suppose that $\{\mathcal{A}_n\}$ is an increasing sequence of algebras, i.e. $\mathcal{A}_n \subset \mathcal{A}_{n+1}$ for all $n \geq 1$. Show that $\cup_{n=1}^{\infty} \mathcal{A}_n$ is an algebra.
(b) Suppose that the $\mathcal{A}_n$ of (a) are $\sigma-$algebras. Show by constructing a counter-example that $\cup_{n=1}^{\infty} \mathcal{A}_n$ need not be a $\sigma-$algebra.

Solution: (a) If $A \in \cup_{n=1}^{\infty} \mathcal{A}_n$, then $A \in \mathcal{A}_m$ for some m, and since $\mathcal{A}_m$ is an algebra, $A^c \in \mathcal{A}_m$. Hence $A^c \in \cup_{n=1}^{\infty} \mathcal{A}_n$. If $A, B \in \cup_{n=1}^{\infty} \mathcal{A}_n$, then $A \in \mathcal{A}_m$ for some m and $B \in \mathcal{A}_n$ for some n. Without loss we can assume that $m \leq n$, and since $\mathcal{A}_m \subset \mathcal{A}_n$ it follows that $A, B \in \mathcal{A}_n$. Since $\mathcal{A}_n$ is an algebra, it follows that $A \cup B \in \mathcal{A}_n$, and hence that $A \cup B \in \cup_{n=1}^{\infty} \mathcal{A}_n$.
(b) Take $\Omega = [0, 1]$. Let $\mathcal{A}_0 = \{\emptyset, \Omega\}$, $\mathcal{A}_2 = \sigma[\mathcal{A}_0, [0, 1/2)]$, ..., $\mathcal{A}_n = \sigma[\mathcal{A}_{n-1}, [0, 1 - 1/n)]$, ... Then $\mathcal{A}_n \subset \mathcal{A}_{n+1}$ by construction, but $\cup_{n=1}^{\infty} \mathcal{A}_n$ is not a sigma field: if we let $A_k = [0, 1 - 1/k)$ for each $k = 1, 2, \ldots$, then $A_k \in \cup_{n=1}^{\infty} \mathcal{A}_n$ since $A_k \in \mathcal{A}_k$ by construction, but $[0, 1) = \cup_{k=1}^{\infty} A_k \notin \cup_{n=1}^{\infty} \mathcal{A}_n$.

Homework #1 Problem #7 Andrada

Let $\mathcal{A} = \{[0,1]\} \cup \{[\frac{1}{2^{n+1}}; \frac{1}{2^n}); n = 0,1 \dots\}$.

Which of the following subsets belong to $\sigma[\mathcal{A}]$:

a) $\{0\}$ b) $\{1\}$ c) $\{\frac{1}{2}\}$ d) $\{\frac{1}{3}\}$

e) $\{0,1\}$ f) $(\frac{1}{4}, 1]$ g) $[0, \frac{1}{2})$ h) $[\frac{1}{4}, 1)$

i) $(0, \frac{1}{2})$?

Let $\tilde{\mathcal{A}} = \{\{0,1\}\} \cup \{[\frac{1}{2^{n+1}}; \frac{1}{2^n}); n = 0,1 \dots\}$

Step 1: Show $\sigma[\mathcal{A}] = \sigma[\tilde{\mathcal{A}}]$. In order to show this we need to prove:

(1) $\mathcal{A} \subset \sigma[\tilde{\mathcal{A}}]$

We know that $\tilde{\mathcal{A}} \subset \sigma[\tilde{\mathcal{A}}]$

$\bigcup_{n=0}^{\infty} [\frac{1}{2^{n+1}}; \frac{1}{2^n}) \in \sigma[\tilde{\mathcal{A}}]$ since $\sigma[\tilde{\mathcal{A}}]$ is closed under countable unions of sets in $\tilde{\mathcal{A}}$.

$(0,1) \in \sigma[\tilde{\mathcal{A}}]$

Notice $\{0,1\} \in \sigma[\tilde{\mathcal{A}}]$ since $\tilde{\mathcal{A}} \subset \sigma[\tilde{\mathcal{A}}]$

Hence $(0,1) \cup \{0,1\} \in \sigma[\tilde{\mathcal{A}}]$

$[0,1] \in \sigma[\tilde{\mathcal{A}}] \Rightarrow \mathcal{A} \subset \sigma[\tilde{\mathcal{A}}]$

(2) $\tilde{\mathcal{A}} \subset \sigma[\mathcal{A}]$

We know that $\mathcal{A} \subset \sigma[\mathcal{A}]$

$\bigcup_{n=0}^{\infty} [\frac{1}{2^{n+1}}; \frac{1}{2^n}) \in \sigma[\mathcal{A}]$ since $\sigma[\mathcal{A}]$ is closed under countable unions of sets in $\mathcal{A}$.

$(0,1) \in \sigma[\mathcal{A}]$

Notice $[0,1] \in \sigma[\mathcal{A}]$ since $\mathcal{A} \subset \sigma[\mathcal{A}]$

Hence $[0,1] \setminus (0,1) \in \sigma[\mathcal{A}]$

$\{0,1\} \in \sigma[\mathcal{A}] \Rightarrow \tilde{\mathcal{A}} \subset \sigma[\mathcal{A}]$

$\overset{(1),(2)}{\Longrightarrow} \sigma[\mathcal{A}] = \sigma[\tilde{\mathcal{A}}]$

Note : $\Omega = [0,1]$

$\mathcal{A}$ is a countable partition of Ω with sets that are not disjoint.

$\tilde{\mathcal{A}}$ is a countable partition of Ω with sets that are disjoint.

$\sigma[\tilde{\mathcal{A}}]$ consist of countable unions of elements of $\tilde{\mathcal{A}}$.

Take:

$$\bigcup_{n=1}^{\infty} \left[\frac{1}{2^{n+1}} ; \frac{1}{2^n}\right) = \underline{(0,1)} \in \sigma[\tilde{\mathcal{A}}]$$

$$\bigcup_{n=0}^{1} \left[\frac{1}{2^{n+1}} ; \frac{1}{2^n}\right) = \underline{\left[\frac{1}{4} ; 1\right)} \in \sigma[\tilde{\mathcal{A}}]$$

Since $\tilde{\mathcal{A}} \subset \sigma[\tilde{\mathcal{A}}] \Rightarrow \underline{\{\{0,1\}\}} \in \sigma[\tilde{\mathcal{A}}]$.

Subsets e), h) and i) belong to $\sigma[\tilde{\mathcal{A}}]$.

The other subsets cannot be written as unions of elements of $\tilde{\mathcal{A}}$, hence do not belong to $\sigma[\tilde{\mathcal{A}}]$.

4. c). Let $\Omega = [0,1]$.

Def : $A_1 = [0, \frac{1}{2}]$, $A_2 = [0, \frac{1}{3}]$, $\cdots$, $A_n = [0, \frac{1}{n+1}]$, $\cdots$

and $\mathcal{C}_1 = \sigma[\{A_1\}]$, $\mathcal{C}_2 = \sigma[\{A_1, A_2\}]$, $\cdots$

$\mathcal{C}_n = \sigma[\{A_1, A_2, \cdots, A_n\}]$, $\cdots$

Now clearly, $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \cdots \subseteq \mathcal{C}_n \subseteq \cdots$

We consider $\bigcup_{i=1}^{\infty} \mathcal{C}_i$. Obviously, $A_n \in \bigcup_{i=1}^{\infty} \mathcal{C}_i$, $\forall n \geq 1$.

Now we want to prove

$\bigcap_{i=1}^{\infty} A_i = \{0\} \notin \bigcup_{i=1}^{\infty} \mathcal{C}_i$ which means $\bigcup_{i=1}^{\infty} \mathcal{C}_i$ is not a σ-field.

Assume $\{0\} \in \bigcup_{i=1}^{\infty} \mathcal{C}_i \Rightarrow \exists n_0 \in \mathbb{N}$, $\{0\} \in \mathcal{C}_{n_0}$.

Let $B_1 = (\frac{1}{2}, 1]$, $B_2 = (\frac{1}{3}, \frac{1}{2}]$, $\cdots$, $B_{n_0} = (\frac{1}{n_0+1}, \frac{1}{n_0}]$, $B_{n_0+1} = [0, \frac{1}{n_0+1}]$.

Clearly $B_1, \cdots, B_{n_0+1}$ is a partition of $[0,1]$.

Def: $\mathcal{D} = \{B_1, \cdots, B_{n_0+1}\}$

Now we have $\boxed{\mathcal{C}_{n_0} = \sigma[\mathcal{D}]}$.

Since $\mathcal{D} \subseteq \mathcal{C}_{n_0} \Rightarrow \sigma[\mathcal{D}] \subseteq \mathcal{C}_{n_0}$.

and $\{A_1, \cdots, A_{n_0}\} \subseteq \sigma[\mathcal{D}] \Rightarrow \mathcal{C}_{n_0} \subseteq \sigma[\mathcal{D}]$

(5)

Now I $\boxed{\text{claim}}$ that any element in $\sigma[\mathcal{D}]$ is expressed as a countable (finite, in this case) union of elements in $\mathcal{D}$.

Let $\mathcal{F}$ = set of all countable union of elements in $\mathcal{D}$.

we need to show $\sigma[\mathcal{D}] = \mathcal{F}$.

Clearly $\mathcal{F} \subseteq \sigma[\mathcal{D}]$ and $\mathcal{D} \subseteq \mathcal{F}$.

Now we prove $\mathcal{F}$ is a σ-field.

(a). $\Omega = [0,1] = \bigcup_{i=1}^{n_0+1} B_i \in \mathcal{F}$.

(b). If $A \in \mathcal{F}$. $\Rightarrow A = \bigcup_{k=1}^{\infty} B_{n_k}$ where

$$B_{n_1}, \cdots, B_{n_k} \cdots \in \mathcal{D} = \{B_1, -, B_{n_0+1}\}.$$

Now clearly $A^c = \bigcup_{k=1}^{\infty} B_{n_k'}$ where $B_{n_k'} \in \mathcal{D}$

and $n_k' \neq n_1, \cdots, n_k$ for any k.

Therefore $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$.

(c). If $A_1, \cdots, A_n \cdots \in \mathcal{F}$. $\Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

since each A_i is a countable union of sets in $\mathcal{D}$

Thus by (a), (b), (c), $\mathcal{F}$ is a σ-field. $\Rightarrow \sigma[\mathcal{D}] = \mathcal{F}$.

If $\{0\} \in \mathcal{C}_{n_0} = \sigma[\mathcal{D}] = \mathcal{F}$, we must have $\{0\}$ can be written as a countable / finite union of B_i, $i=1, \cdots n_0+1$

which is impossible. Contradict! $\Rightarrow \{0\} \notin \mathcal{C}_{n_0}$ ⊠

(6)

5. Let $\mathcal{F} = \bigcap_{j \in \mathbb{R}} \mathcal{F}_j$. To show $\mathcal{F}$ is a σ-field, we need to prove:

① $\Omega \in \mathcal{F}$. ② If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$.

③ If $A_1, \cdots, A_n, \cdots \in \mathcal{F}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

Now ① is obvious since $\Omega \in \mathcal{F}_j$ for all j.

② If $A \in \mathcal{F}$, we have $A \in \mathcal{F}_j$ for all j.

then $A^c \in \mathcal{F}_j$ for all j. $\Rightarrow A^c \in \mathcal{F}$.

③. If $A_1, \cdots A_n, \cdots \in \mathcal{F}$, then $A_1, \cdots, A_n, \cdots \in \mathcal{F}_j$ all j

$\Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}_j$ all j $\Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

6. We need to prove

① $\sigma[\mathcal{A}] \subset \tilde{\mathcal{F}}$ and ② $\tilde{\mathcal{F}} \subset \sigma[\mathcal{A}]$.

For ①, since $\mathcal{A} \subset \tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}$ is a σ-field, we have

$\sigma[\mathcal{A}] = \bigcap \{\mathcal{F} : \mathcal{F}$ is a σ-field such that $\mathcal{A} \subset \mathcal{F}\}$

$\subset \tilde{\mathcal{F}}$

For ②, since $\sigma[\mathcal{A}]$ is a σ-field and $\mathcal{A} \in \sigma[\mathcal{A}]$

$\Rightarrow \tilde{\mathcal{F}} \subset \sigma[\mathcal{A}]$.