Class : set of sets

General Stuff

To show 2 sets A & B are equal $\longrightarrow$ show $\left.\begin{array}{l} A \subseteq B \\ B \subseteq A \end{array}\right\} \Rightarrow A \equiv B$

#1 Family of all subsets of Ω = powerset of Ω denoted by $2^{\Omega} = \mathcal{C}$

We have to show that the power set is a σ-field

We have to show (i) $\Omega \in \mathcal{C}$

a class $\mathcal{C}$ is a σ-field (ii) $A \in \mathcal{C} \Rightarrow A^c \in \mathcal{C}$

(iii) $A_1, A_2 \dots \in \mathcal{C} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{C}$

(i) 2^{Ω} = set of all subsets so trivially $\Omega \in \mathcal{C}$

(ii) $A \in \mathcal{C} \Rightarrow A^c \in \mathcal{C}$ since $A^c \subseteq \Omega$

(iii) $A_1, A_2 \dots \in \mathcal{C}$, Now $A_1 \dots A_i \subseteq \Omega \Rightarrow \bigcup_{i=1}^{\infty} A_i \subseteq \Omega$

$$\Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{C}$$

#3 $\mathcal{F}$ is a σ-field on $\Omega = [0,1]$ st $\left[\frac{1}{n+1}, \frac{1}{n}\right] \in \mathcal{F} \quad \forall n \geq$

We know $\mathcal{F}$ is a σ-field $\Rightarrow$ (i) $\Omega \in \mathcal{F}$ ie $[0,1] \in \mathcal{F}$

(ii) If $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

(iii) $A_1, A_2 \dots \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

(a) Since $\left[\frac{1}{n+1}, \frac{1}{n}\right] \in \mathcal{F} \quad \forall n \Rightarrow \bigcup_{n=1}^{\infty} \left[\frac{1}{n+1}, \frac{1}{n}\right] \in \mathcal{F}$

Now $\bigcup_{n=1}^{\infty} \left[\frac{1}{n+1}, \frac{1}{n}\right] = (0,1] \in \mathcal{F}$ —— (*)

$\therefore \quad (0,1]^c \in \mathcal{F}$

ie $\{0\} \in \mathcal{F}$

(b) $\left[\frac{1}{n+1}, \frac{1}{n}\right] \in \mathcal{F} \quad \forall\, n \geq 1$

$\therefore \left[\frac{1}{n}, \frac{1}{n-1}\right] \in \mathcal{F}$ also $\quad \forall\, n \geq 2$.

$\Rightarrow \left[\frac{1}{n}, \frac{1}{n-1}\right] \cap \left[\frac{1}{n+1}, \frac{1}{n}\right] \in \mathcal{F}$

$\Rightarrow \{1/n\} \in \mathcal{F} \quad \forall\, n \geq 2$.

$\Rightarrow \bigcup_{n=2}^{\infty} \{1/n\} \in \mathcal{F}$

$\Rightarrow \{1/n,\ n = 2, 3, 4 \ldots\} \in \mathcal{F}$.

(c) We know $\left[\frac{1}{n}, \frac{1}{n-1}\right], \left[\frac{1}{n-1}, \frac{1}{n-2}\right] \cdots \left[\frac{1}{2}, 1\right] \in \mathcal{F}$

$\Rightarrow \bigcup_{k=2}^{n} \left[\frac{1}{k}, \frac{1}{k-1}\right] \in \mathcal{F}$

$$\bigcup_{k=2}^{n} \left[1/k, \frac{1}{k-1}\right] = \left[\frac{1}{n}, 1\right] \in \mathcal{F}$$

We also know $\{1/n\} \in \mathcal{F}$

$$\left[\frac{1}{n}, 1\right] \cap \{1/n\}^c = (1/n, 1] \in \mathcal{F}$$

(d) We know $(1/n, 1] \in \mathcal{F}$ & $(0, 1] \in \mathcal{F}$ [from (*) in (a)]

$(0, 1] \setminus (1/n, 1] = (0, 1/n] \in \mathcal{F}$

#5] $\{\mathcal{F}_j\}_{j \in \mathcal{J}}$ is a collection of σ-fields defined on the same set Ω

To show $\mathcal{A} = \bigcap_{j \in \mathcal{J}} \mathcal{F}_j$ is also a σ-field

(i) We know $\mathcal{F}_j$ is a σ-field on $\Omega \Rightarrow \Omega \in \mathcal{F}_j \quad \forall\, j \in \mathcal{J}$

$\therefore \Omega \in \bigcap_{j \in \mathcal{J}} \mathcal{F}_j$

(ii) $A \in \bigcap_{j \in \mathcal{J}} \mathcal{F}_j \implies A \in \mathcal{F}_j \quad \forall\, j \in \mathcal{J}$

$\rightarrow A^c \in \mathcal{F}_j \quad \forall\, j \in \mathcal{J}$

$\implies A^c \in \bigcap_{j \in \mathcal{J}} \mathcal{F}_j$

(iii). $A_1, A_2 \dots \in \mathcal{A} = \bigcap_{j \in \mathcal{J}} \mathcal{F}_j \implies A_1, A_2 \dots \in \mathcal{F}_j \quad \forall\, j \in \mathcal{J}$

$\implies \bigcup_{k=1}^{\infty} A_k \in \mathcal{F}_j \quad \forall\, j \in \mathcal{J}$

$\implies \bigcup_{k=1}^{\infty} A_k \in \left[\bigcap_{j \in \mathcal{J}} \mathcal{F}_j\right]$

#2 Aside : A set whose complement is finite is called a cofinite s
To check if $\mathcal{F}$ = set of all finite & cofinite subsets of Ω is always a σ-field

(i) $\phi \in \mathcal{F}$ (by def) $\therefore \phi^c = \Omega \in \mathcal{F}$ (by def) ✓a
(ii) $A \in \mathcal{F}$ then either A is finite or A is cofinite
If A is finite $\implies A^c$ is cofinite since $(A^c)^c = A$ which is fi
$\therefore A^c \in \mathcal{F}$
If A is cofinite, by def A^c is finite $\implies A^c \in \mathcal{F}$ ✓al

(iii) If Ω is finite $\Rightarrow$ all subsets $A \subseteq \Omega$ will be finite
$\implies \bigcup_{i=1}^{\infty} A_i$ will be finite $\implies \in \mathcal{F}$

<u>If Ω is not finite</u>
Find an example where $\bigcup_{i=1}^{\infty} A_i$ is neither finite or cofinite

4. $\{\mathcal{A}_n\}_n$ is an $\uparrow$ seq of fields ie $\mathcal{A}_n \subset \mathcal{A}_{n+1}$ $\forall$ $n \geq 1$

Show $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{A}_n$ is a field

(i) $\Omega \in \mathcal{A}_n$ $\forall$ $n \geq 1$ since $\mathcal{A}_n$ is a field

$\therefore$ $\Omega \in \bigcup_{n=1}^{\infty} \mathcal{A}_n$

(ii) $A \in \mathcal{B}$ ie $A \in \bigcup_{n=1}^{\infty} \mathcal{A}_n$ $\Rightarrow$ $\exists$ k st $A \in \mathcal{A}_k$

$\Rightarrow$ $A^c \in \mathcal{A}_k$ for some $k \geq 1$

$\Rightarrow$ $A^c \in \bigcup_{n=1}^{\infty} \mathcal{A}_n$

(iii) $A_1, A_2, A_3 \dots A_n \in \mathcal{B}$ ie $A_1, A_2 \text{--} A_n \in \bigcup_{n=1}^{\infty} \mathcal{A}_n$

$\Rightarrow$ $\exists$ some $k_1, k_2 \dots k_n$ st $A_1 \in \mathcal{A}_{k_1}$

$A_2 \in \mathcal{A}_{k_2}$

$A_3 \in \mathcal{A}_{k_3}$ ....

Let $m = \max\{k_1, k_2 \dots k_n\}$

$\therefore$ $\mathcal{A}_m = \bigcup_{i=1}^{n} \mathcal{A}_{k_i}$ since $\{\mathcal{A}_n\}$ is $\uparrow$ seq of fields

$\Rightarrow$ $A_1 \in \mathcal{A}_m$, $A_2 \in \mathcal{A}_m$ ... $A_n \in \mathcal{A}_m$ also

$\Rightarrow$ $\bigcup_{i=1}^{n} A_i \in \mathcal{A}_m$

$\Rightarrow$ $\bigcup_{i=1}^{n} A_i \in \bigcup_{n=1}^{\infty} \mathcal{A}_n$ $\{$since $\bigcup_{n=1}^{\infty} \mathcal{A}_n$ includes $\mathcal{A}_m$

$A = \bigcap_k B_k \Rightarrow A \subset B_k \quad \forall k$

4(b) $\Omega = \{0,1,2,3\}$

$\mathcal{F}_1 = \{\Omega, \{1\}, \{0,2,3\}, \phi\}$

$\mathcal{F}_2 = \{\Omega, \{0\}, \{1,2,3\}, \phi\}$

$\mathcal{F}_1 \cup \mathcal{F}_2 = \{\Omega, \{1\}, \{0\}, \{1\,2\,3\}, \{0,2,3\}, \phi\}$

which is not a field since it is not closed unde[r] union

(c) Find counter example

#6 $\sigma[\mathcal{A}] = \bigcap_j \{\mathcal{F}_j : \mathcal{A} \subseteq \mathcal{F} \text{ \& } \mathcal{F} \text{ is a } \sigma\text{-field}\}$

$\tilde{\mathcal{F}}$ is a σ-field st (a) $\mathcal{A} \subset \tilde{\mathcal{F}}$

(b) If $\mathcal{F}$ is a σ-field & $\mathcal{A} \subset \mathcal{F} \Rightarrow \tilde{\mathcal{F}} \subset \mathcal{F}$

We know $\mathcal{A} \overset{\text{by definition}}{\subseteq} \underbrace{\sigma[\mathcal{A}]}_{\text{is a } \sigma\text{-field}} \Longrightarrow \tilde{\mathcal{F}} \subseteq \sigma[\mathcal{A}]$

Think ?? ok!

$\tilde{\mathcal{F}}$ is a σ-field and $\mathcal{A} \subset \tilde{\mathcal{F}} \Rightarrow \tilde{\mathcal{F}}$ is one element of $\cap\{\mathcal{F}_j : \}$

$\Rightarrow \sigma[\mathcal{A}] \subseteq \tilde{\mathcal{F}}$

#8 (a) If $\mathcal{C}_1 \subset \mathcal{C}_2 \Rightarrow \sigma[\mathcal{C}_1] \subset \sigma[\mathcal{C}_2]$

$\sigma[\mathcal{C}_2]$ contains all elem of $\mathcal{C}_2$ & thier countable unions

$\therefore \mathcal{C}_1 \subset \sigma[\mathcal{C}_2]$

$\sigma[\mathcal{A}] = \bigcap\{\mathcal{F} : \mathcal{F} \text{ is } \sigma\text{-field \& } \mathcal{A} \subset \mathcal{F}\}$ def

$\therefore \sigma[\mathcal{C}_1] = \cap\{\mathcal{F} : \mathcal{F} \text{ is } \sigma\text{-field \& } \mathcal{C}_1 \in \mathcal{F}\}$

If $A = \bigcap_{\alpha \in \Theta} B_\alpha \Rightarrow A \subset B_\alpha \quad \forall \alpha.$

$\sigma[\mathcal{C}_2]$ is σ-field & $\mathcal{C}_1 \subset \sigma[\mathcal{C}_2]$

$\therefore \quad \sigma[\mathcal{C}_1] \subset \sigma[\mathcal{C}_2]$

8 (b) $\mathcal{C}_1 \subset \sigma[\mathcal{C}_2]$ & $\mathcal{C}_2 \subset \sigma[\mathcal{C}_1]$ then $\sigma[\mathcal{C}_1] = \sigma[\mathcal{C}_2]$

$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$

$\sigma[\mathcal{C}_1] \subset \sigma[\mathcal{C}_2] \qquad \sigma[\mathcal{C}_2] \subset \sigma[\mathcal{C}_1]$ from previous part.

7 Let $\mathcal{A} = \{[0,1]\} \cup \left\{\left[\frac{1}{2^{n+1}}, \frac{1}{2^n}\right) : n = 0, 1, 2 \dots\right\}$

$\tilde{\mathcal{A}} = \{0, 1\} \cup \left\{\left[\frac{1}{2^{n+1}}, \frac{1}{2^n}\right) : n = 0, 1, 2 \dots\right\}$

Step 1 Show $\sigma[\tilde{\mathcal{A}}] = \sigma[\mathcal{A}]$

Step 2 Note all sets in $\tilde{\mathcal{A}}$ are disjoint sets.

If we have a set $\mathcal{C}$ which consists of countable disjoint partition of Ω then $\sigma[\mathcal{C}]$ consists only of countable union of elements of $\mathcal{C}$

Step 3

Any set which can be expressed as a countable union alone of elements of $\tilde{\mathcal{A}}$ then the set is in $\sigma[\tilde{\mathcal{A}}]$

Step 1: To show $\mathcal{A} \subset \sigma[\tilde{\mathcal{A}}]$ and $\tilde{\mathcal{A}} \subset \sigma[\mathcal{A}]$

To show $\tilde{\mathcal{A}} \subset \sigma[\mathcal{A}]$ it is enough to show $\{0,1\} \in \sigma[\mathcal{A}]$

Now $\bigcup_{n=0}^{\infty} \left[\frac{1}{2^{n+1}}, \frac{1}{2^n}\right) = (0, 1)$ (Fact – not trivial!)

$$\left.\begin{array}{l} \therefore \; (0,1) \in \sigma[\mathcal{A}] \\ \text{Also } [0,1] \in \sigma[\mathcal{A}] \end{array}\right\} \Rightarrow [0,1] \setminus (0,1) = \{0,1\} \in \sigma[\mathcal{A}]$$

Also $\left[\frac{1}{2^{n+1}}, \frac{1}{2^n}\right)$ all trivially $\in \sigma[\mathcal{A}]$ $\forall\ n = 0, 1, 2 \ldots$

$\therefore\ \tilde{\mathcal{A}} \in \sigma[\mathcal{A}]$

Now $\bigcup_{n=0}^{\infty} \left[\frac{1}{2^{n+1}}, \frac{1}{2^n}\right) = (0,1) \in \sigma[\tilde{\mathcal{A}}]$

We know $\{0,1\} \subset \sigma[\tilde{\mathcal{A}}]$

$\therefore\ \{0,1\} \cup (0,1) = [0,1] \subset \sigma[\tilde{\mathcal{A}}]$

$\therefore\ \mathcal{A} \subset \sigma[\tilde{\mathcal{A}}]$

Show Step 2 also!

#9] To show $\lambda(C) = 0$ it is enough to show $\lambda(C^c) = 1$ since C is defined on $[0,1]$ which has measure 1.

$C = \bigcap_{n=1}^{\infty} K_n$ where K_n are middle $(1/3)^{rd}$

$C^c = \bigcup_{n=1}^{\infty} K_n^c$

$\lambda(C^c) = \lambda\left(\bigcup_{n=1}^{\infty} K_n^c\right)$ K_n^c is a $\uparrow$ seq $\Rightarrow \bigcup_{n=1}^{N} K_n^c =$

sum $= \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$