STA 5447 – Spring 2006

Hwk 1 : due Tue, Jan the 24th

These results hold even when $P(\Omega) = Q(\Omega)$ and P and Q are any finite measures.

Ex 1 Let P, Q be two probability measures on $(\Omega, \mathcal{A})$ which are both a.c. wrt μ, with R–N derivatives p and q, respectively.

Show that:

a) $d_{TV}(P,Q) \equiv \sup_{A \in \mathcal{A}} |P(A) - Q(A)| = \frac{1}{2} \int |p-q| \, d\mu.$

$=$ [redacted] $\int (p-q)^+ \, d\mu.$

b) $d_{TV}(P,Q)$ does <u>NOT</u> depend on the choice of μ.

- $d_{TV}(P,Q)$ is called the total variation distance between two measures.

Results in density estimation

What matters is the rate of convergence.

eg $a_n = \dfrac{1}{\log(\log n)} \longrightarrow 0$ but you need a very large 'n'

Say we have a rv X with meas μ

Assume μ has a density wrt λ ; $f(x)$ [ie $\mu << \lambda$]

$X_1, \dots X_n$ iid as X

Estimate f.

Let f_n be an estimate of f. (many techniques)

1) You can talk about the quality of f_n in terms of the distance b/w these 2 funct^{n} f_n and f

2) Suppose we want to compute $\int_A f(x)\,dx = P(A)$

To do this we compute $\hat{P}(A) = \int_A f_n(x)\,dx$

Ex 1 : talks about the total variation distance

$$d_{TV}(P, \hat{P}) \equiv \sup_{A \in \mathcal{A}} |P(A) - \hat{P}(A)| = \frac{1}{2}\int |f - f_n|\,dx$$

Ex 2

$$f_n \xrightarrow{a.e} f_0 \Rightarrow d_{VT}(P_n, P_0) \xrightarrow{a.e} 0$$

We can be given $(f_n - f_0)^+ \xrightarrow{a.e} 0$ (weaker then in ques) the result still holds.

No proof

Ex 2 (Scheffe's theorem). Uses Ex 1

Let $f_0, f_1, \ldots, f_n, \ldots$ be positive, defined on $(\Omega, \mathcal{A}, \mu)$ and $\int_\Omega f_n \, d\mu = 1$ for all $n \geq 0$.

a) Assume $f_n \xrightarrow{a.e.} f_0$. Show that

$$(*) \quad \sup_{A \in \mathcal{A}} \left| \int_A f_n \, d\mu - \int_A f_0 \, d\mu \right| \xrightarrow[n \to \infty]{} 0.$$

b) Assume now that $f_n \xrightarrow{\mu} f_0$ and
$$\int_\Omega f_n \, d\mu \longrightarrow \int_\Omega f \, d\mu.$$
given

Show that $(*)$ continues to hold.

Here we do not assume $\int_\Omega f_n \, d\mu = 1$

Ex 3 Let $X_n \sim Bin(n, p_n)$, with $n p_n \xrightarrow[n \to \infty]{} \lambda > 0$.
Let P_n be the induced distribution of X_n on $\mathbb{R}$.
Let $X_0 \sim Poisson(\lambda)$ and let P_0 be its induced dist. on $\mathbb{R}$.
Show that $d_{TV}(P_n, P_0) \longrightarrow 0$ as $n \to \infty$.

$X_n \xrightarrow{d} X_0 \equiv$ Pois

Hwk presentation
1) Ex 1, a) and Ex 2 a)
2) Ex 2 b)
3) Ex 1 b) and Ex 3.

$$\sup_{A \in \mathcal{A}} \left| \int_A f_n - \int_A f_0 \right| = \int |f_n - f_0| \cdot d\mu .$$

$$|f_n - f_0| \leq |f_n| + |f_0|$$
$$= f_n + f_0$$

And $\int (f_n + f_0) d\mu = \int f_n d\mu + \int f_0 d\mu = 2 \quad \in \mathcal{L}_1$

But for the DCT we need a funct$^{\underline{n}}$ $Y \in \mathcal{L}_1$ which is indep of 'n'

We would need some additional assumption on f_n like $f_n \leq f_0$

<u>Conclusion</u> We also have $f_n \geq 0$, $\int f_n = 1 \quad \forall \; n \geq 0$

Assume $\int (f_n - f_0)^+ \xrightarrow{a.e} 0$

Then $d_{VT}(P_n, P_0) = \sup_{A \in \mathcal{A}} \left| \int_A f_n - \int_A f_0 \right| \longrightarrow 0$

1) P & Q are 2 prob meas. on $(\Omega, \mathcal{A})$ both a.c wrt μ. with RN derivatives p, q.

To show $d_{TV}(P,Q) = \sup_{A\in\mathcal{A}} |P(A)-Q(A)| = \frac{1}{2}\int |p-q|\,d\mu = \int (p-q)^+ d\mu.$

$p = \frac{dP}{d\mu}$ and $q = \frac{dQ}{d\mu}$ $\rightarrow$ $\int_\Omega p\,d\mu = P(\Omega) = 1 = Q(\Omega) = \int_\Omega q\,d\mu$

$$\Rightarrow \int (p-q)\,d\mu = 0$$

$$\Rightarrow \int [(p-q)^+ - (p-q)^-]\,d\mu = 0$$

$$\Rightarrow \int (p-q)^+ d\mu = \int (p-q)^- d\mu$$

Now $|p-q| = (p-q)^+ + (p-q)^-$

$$\int |p-q|\,d\mu = \int (p-q)^+ d\mu + \int (p-q)^- d\mu$$

$$= 2\int (p-q)^+ d\mu$$

$\therefore$

$$\frac{1}{2}\int |p-q|\,d\mu = \int (p-q)^+ d\mu.$$

Now
$$\begin{aligned}
2|P(A)-Q(A)| &= |P(A)-Q(A)| + |P(A)-Q(A)| \\
&= |P(A)-Q(A)| + |P(A)-1+1-Q(A)| \\
&= |P(A)-Q(A)| + |P(A^c)-Q(A^c)| \\
&= \left|\int_A (p-q)\,d\mu\right| + \left|\int_{A^c} (p-q)\,d\mu\right| \\
&\le \int_A |p-q|\,d\mu + \int_{A^c} |p-q|\,d\mu = \int_\Omega |p-q|\,d\mu
\end{aligned}$$

$$\sup_{A\in\mathcal{A}} 2|P(A)-Q(A)| \leq \int_{\Omega} |p-q|\,d\mu = 2\int (p-q)^{+}\,d\mu \qquad \text{—}(*)$$

Let $A^{\circ} \equiv \{(p-q) \geq 0\}$

$$\sup_{A\in\mathcal{A}} 2|P(A)-Q(A)| \geq 2|P(A^{\circ})-Q(A^{\circ})| \qquad \forall A^{\circ}\in\mathcal{A}$$

$$= 2\left|\int_{A^{\circ}} (p-q)\,d\mu\right|$$

$$= 2\left|\int_{\{p-q\geq 0\}} (p-q)\,d\mu\right|$$

$$= 2\left|\int (p-q)^{+}\,d\mu\right|$$

$$= 2\int (p-q)^{+}\,d\mu$$

$$\therefore \quad \sup_{A\in\mathcal{A}} 2|P(A)-Q(A)| \geq 2\int (p-q)^{+}\,d\mu \qquad \text{—}(**)$$

$(*)$ & $(**)$ $\Rightarrow$ $$\sup_{A\in\mathcal{A}} 2|P(A)-Q(A)| = 2\int (p-q)^{+}\,d\mu$$

$$\Rightarrow \sup_{A\in\mathcal{A}} |P(A)-Q(A)| = \int (p-q)^{+}\,d\mu = \frac{1}{2}\int |p-q|\,d\mu$$

1(b) To show $d_{TV}(P,Q)$ does not depend on the choice of μ

Assume there are Lebesgue measures μ_1 and μ_2 st

$P, Q << \mu_1$ with RN derivatives p_1, q_1

$P, Q << \mu_2$ with RN derivatives p_2, q_2

$$d_{TV}(P,Q) \equiv \sup_{A\in\mathcal{A}} |P(A)-Q(A)| = \frac{1}{2}\int |p_1-q_1|\,d\mu_1 = \frac{1}{2}\int |p_2-q_2|\,d\mu_2 \ \{\text{by def}$$

$\therefore$ the choice of μ does not matter.

2] Scheffé's Theorem: $f_0, f_1, \cdots f_n \cdots$ are positive, defined on $(\Omega, \mathcal{A}, \mu)$

with $\int_\Omega f_n \, d\mu = 1 \quad \forall\, n \geq 0$

Given $f_n \xrightarrow{a.e} f_0$ to show $\sup_{A \in \mathcal{A}} \left| \int_A f_n \, d\mu - \int_A f_0 \, d\mu \right| \longrightarrow 0$

Now $\left| \int_A f_n \, d\mu - \int_A f_0 \, d\mu \right| = \left| \int_A (f_n - f_0) \, d\mu \right|$

$$\leq \int_A |f_n - f_0| \, d\mu$$

$$\leq \int_\Omega |f_n - f_0| \, d\mu$$

$$\sup_{A \in \mathcal{A}} \left| \int_A f_n \, d\mu - \int_A f_0 \, d\mu \right| \leq \int_\Omega |f_n - f_0| \, d\mu$$

Now $f_n \xrightarrow{a.e} f_0 \Rightarrow f_n - f_0 \xrightarrow{a.e} 0$

$\Rightarrow |f_n - f_0| \xrightarrow{a.e} 0 \quad \Rightarrow (f_n - f_0)^+ \xrightarrow{a.e} 0$

Showing $\int |f_n - f_0| \, d\mu \longrightarrow 0$ is equivalent to showing $2\int (f_0 - f_n)^+ d\mu \longrightarrow 0$

Also $\left. \begin{array}{l} (f_0 - f_n)^+ \leq f_0 \in \mathcal{L}_1 \\ (f_n - f_0)^+ \xrightarrow{a.e} 0 \end{array} \right\} \Rightarrow \int (f_0 - f_n)^+ d\mu \longrightarrow 0 \quad \{$by DCT

$$\Rightarrow 2 \int (f_0 - f_n)^+ d\mu \longrightarrow 0$$

$\therefore \sup_{A \in \mathcal{A}} \left| \int_A f_n \, d\mu - \int_A f_0 \, d\mu \right| \longrightarrow 0$

2 b) $f_0, f_1, \dots f_n \geq 0$ defined on $(\Omega, \mathcal{A}, \mu)$ with $\int_\Omega f_0 \, d\mu = 1$

$f_n \xrightarrow{\mu} f_0$ and $\int_\Omega f_n \, d\mu \longrightarrow \int_\Omega f_0 \, d\mu = 1$ (given)

Show $\sup_{A \in \mathcal{A}} \left| \int_A f_n \, d\mu - \int_A f_0 \, d\mu \right| \longrightarrow 0$

$\forall A \in \mathcal{A}$

$$\left| \int_A f_n \, d\mu - \int_A f_0 \, d\mu \right| = \left| \int_A (f_n - f_0) \, d\mu \right|$$

$$\leq \int_A |f_n - f_0| \, d\mu$$

$$\leq \int_\Omega |f_n - f_0| \, d\mu$$

We use Vitali's theorem: If $X_n \in \mathcal{L}_r$ and $X_n \xrightarrow{\mu} X$ then

$X_n \xrightarrow{r} X \iff E|X_n|^r \longrightarrow E|X|$

ie $X_n \xrightarrow{\mu} X \iff \int |X_n|^r \, d\mu \longrightarrow \int |X| \, d\mu$

Given $\int f_n \, d\mu \longrightarrow \int f_0 \, d\mu \Rightarrow \forall \varepsilon > 0 \ \exists N_\varepsilon$ st $\left| \int f_n \, d\mu - \int f_0 \, d\mu \right| \leq \varepsilon \quad \forall n \geq N_\varepsilon$

$$\Rightarrow \int f_0 \, d\mu - \varepsilon \leq \int f_n \, d\mu \leq \int f_0 \, d\mu + \varepsilon$$

$\int f_0 \, d\mu \pm \varepsilon$ are both $< \infty \Rightarrow \int f_n \, d\mu < \infty$

$\Rightarrow f_n \in \mathcal{L}_1$

We now show $E|f_n| \longrightarrow E|f_0|$ which will $\Rightarrow f_n \xrightarrow{\mathcal{L}_1} f_0$ (by Vitali's)

$$E|f_n| = \int |f_n| \, d\mu = \int f_n \, d\mu \qquad \{ f_n\text{'s} \geq 0$$

$$E|f_0| = \int |f_0| \, d\mu = \int f_0 \, d\mu \qquad \{ f_0 \geq 0$$

$\therefore$ $E|f_n| \longrightarrow E|f_0|$ $\{$since we are given $\int f_n \, d\mu \longrightarrow \int f_0 \, d\mu$

$\Rightarrow$ $f_n \xrightarrow{L_1} f_0$

$\Rightarrow$ $E|f_n - f_0| \longrightarrow 0$

ie $\int |f_n - f_0| \, d\mu \longrightarrow 0$

$\therefore$ $\left| \int_A f_n \, d\mu - \int_A f_0 \, d\mu \right| \leq \int |f_n - f_0| \, d\mu \longrightarrow 0$

$\therefore$ $\sup_{A \in \mathcal{A}} \left| \int_A f_n \, d\mu - \int_A f_0 \, d\mu \right| \longrightarrow 0$

3] $X_n \sim Bin(n, p_n)$ with $np_n \longrightarrow \lambda > 0$ as $n \to \infty$. [P_n is induced meas]
$X_0 \sim Pois(\lambda)$ [P_0 is the induced m]

To show $d_{TV}(P_n, P_0) \longrightarrow 0$ as $n \to \infty$.

Proof: P_n and P_0 are both $<< \mu$
where μ is the counting meas on $\mathbb{Z}$

$$\int_A P_n \, d\mu = \sum_{\# \text{ integ in } A} P_n$$

If we can show $P_n \xrightarrow{a.e} P_0$ then by Sheffe's theorem

$$d_{TV}(P_n, P_0) \longrightarrow 0$$

$$P_n(k) = \binom{n}{k} p_n^k (1-p_n)^{n-k}$$

$$= \frac{n!}{(n-k)!\, k!} p_n^k (1-p_n)^{n-k}$$

$$= \frac{n(n-1)\cdots(n-k+1)}{k!} p_n^k (1-p_n)^{n-k}$$

$$= \frac{n^k \, 1(1-1/n)(1-2/n)\cdots(1-(k+1/n))}{k!} p_n^k (1-p_n)^{n-k}$$

$$= \frac{(np_n)^k}{k!} (1-p_n)^{n-k} \quad 1(1-1/n)\cdots\cdots(1-\tfrac{k-1}{n})$$

$$\frac{(np_n)^k}{k!} \longrightarrow \frac{\lambda^k}{k!}$$

$$(1-p_n)^{n-k} = \frac{\left[(1-p_n)^{1/p_n}\right]^{np_n}}{(1-p_n)^k} \longrightarrow (e^{-1})^{\lambda} = e^{-\lambda}$$

$$1(1-1/n)(1-2/n)\cdots\cdots(1-k-1/n) \longrightarrow 1$$

$$\therefore \quad P_n(k) \longrightarrow \frac{e^{-\lambda}\lambda^k}{k!} \equiv P_0(k)$$

$$\therefore \quad P_n \longrightarrow P_0$$

QED

HW # 1

Jan 24, 2006

QN 1 (a) Given

P and Q are two probability measures on the same measurable space $(\Omega, \mathcal{A})$. P, Q both a.c. w.r.to measure μ with densities (Radon-Nikodyn derivatives) p and q respectively

$\therefore$ $P(A) = \int_A p\, d\mu$ and $Q(A) = \int_A q\, d\mu$ for $A \in \mathcal{A}$

To show: $d_{TV}(P,Q) = \sup_{A \in \mathcal{A}} |P(A) - Q(A)| = \frac{1}{2}\int |p-q|\, d\mu = \int (p-q)^+ d\mu$

Proof:- As p and q are densities we can write

$$\int_\Omega p\, d\mu = 1 = \int_\Omega q\, d\mu$$

$$\Rightarrow \int_\Omega (p-q)\, d\mu = 0$$

$$\Rightarrow \int_\Omega [(p-q)^+ - (p-q)^-]\, d\mu = 0$$

$$\Rightarrow \int_\Omega (p-q)^+ d\mu = \int (p-q)^- d\mu$$

Now, $|p-q| = (p-q)^+ + (p-q)^-$

$$\Rightarrow \int |p-q|\, d\mu = \int (p-q)^+ d\mu + \int (p-q)^- d\mu = 2\int (p-q)^+ d\mu$$

$$\Rightarrow \frac{1}{2}\int |p-q|\, d\mu = \int (p-q)^+ d\mu \quad \text{——— } (1)$$

Again,
$$\begin{aligned}
2|P(A) - Q(A)| &= |P(A) - Q(A)| + |P(A) - Q(A)| \\
&= |P(A) - Q(A)| + |1 - Q(A) - (1 - P(A))| \\
&= |P(A) - Q(A)| + |Q(A^c) - P(A^c)| \\
&= |P(A) - Q(A)| + |P(A^c) - Q(A^c)| \\
&= \left|\int_A (p-q)\, d\mu\right| + \left|\int_{A^c} (p-q)\, d\mu\right| \\
&\le \int_A |p-q|\, d\mu + \int_{A^c} |p-q|\, d\mu = \int_\Omega |p-q|\, d\mu
\end{aligned}$$

$$\Rightarrow \sup_{A \in \mathcal{A}} 2|P(A) - Q(A)| \le \int_\Omega |p-q|\, d\mu \quad (*)$$

Next, Let $A^\circ = \{(p-q) \ge 0\}$

We know

$$\sup_{A \in \mathcal{A}} 2\,|P(A) - Q(A)| \geq 2\,|P(A^0) - Q(A^0)| \quad \forall A^0 \in \mathcal{A}$$

$$= 2\left| \int_{A^0} (p-q)\, d\mu \right|$$

$$= 2\left| \int_{\{(p-q)\geq 0\}} (p-q)\, d\mu \right|$$

$$= 2\left| \int (p-q)\, 1_{\{(p-q)\geq 0\}}\, d\mu \right|$$

$$= 2\left| \int (p-q)^+\, d\mu \right|$$

$$= 2 \int (p-q)^+\, d\mu \qquad [\because (p-q)^+ \geq 0]$$

$$= \int |p-q|\, d\mu$$

$$\Rightarrow \sup_{A \in \mathcal{A}} 2\,|P(A) - Q(A)| \geq \int |p-q|\, d\mu \qquad (**)$$

From (*) and (**),

$$\sup_{A \in \mathcal{A}} 2\,|P(A) - Q(A)| = \int |p-q|\, d\mu$$

$$\Rightarrow \sup_{A \in \mathcal{A}} |P(A) - Q(A)| = \frac{1}{2}\int |p-q|\, d\mu \qquad \text{—— (2)}$$

From (1) and (2)

$$d_{TV}(P, Q) = \sup_{A \in \mathcal{A}} |P(A) - Q(A)| = \frac{1}{2}\int |p-q|\, d\mu = \int (p-q)^+\, d\mu.$$

q.e.d.

QN 2(a)

Given

$f_0, f_1, \cdots f_n$ are positive defined on $(\Omega, \mathcal{A}, \mu)$ and $\int_\Omega f_n \, d\mu = 1$ $\forall n \geq 0$

$$f_n \xrightarrow{a.e.} f_0$$

To show

$$\sup_{A \in \mathcal{A}} \left| \int_A f_n \, d\mu - \int_A f_0 \, d\mu \right| \xrightarrow[n\to\infty]{} 0$$

Proof :- $\left| \int_A f_n \, d\mu - \int_A f_0 \, d\mu \right| = \left| \int_A (f_n - f_0) \, d\mu \right| \leq \int_A |f_n - f_0| \, d\mu \leq \int_\Omega |f_n - f_0| \, d\mu$

$$\therefore \sup_{A \in \mathcal{A}} \left| \int_A f_n \, d\mu - \int_A f_0 \, d\mu \right| \leq \int_\Omega |f_n - f_0| \, d\mu \quad \text{---} \quad (*)$$

Given $f_n \xrightarrow{a.e.} f_0$

$\Rightarrow f_n - f_0 \xrightarrow{a.e.} 0$

$\Rightarrow |f_n - f_0| \xrightarrow{a.e.} 0$

Now, to show $\int |f_n - f_0| \, d\mu \to 0$ is equivalent to show

$$2 \int (f_0 - f_n)^+ \, d\mu \longrightarrow 0 \quad \text{-------} \quad (**)$$

we know, $0 \leq (f_0 - f_n)^+ \leq |f_0 - f_n|$

$$\left.\begin{array}{l} \text{as } |f_0 - f_n| \xrightarrow{a.e.} 0 \\ \Rightarrow (f_0 - f_n)^+ \xrightarrow{a.e.} 0 \\ \text{Again, } (f_0 - f_n)^+ \leq f_0 \in \mathcal{L}_1 \end{array}\right\} \text{DCT}$$

$\therefore$ By DCT, $\int [f_0 - f_n]^+ \, d\mu \xrightarrow[n\to\infty]{} 0$

$$\Rightarrow 2 \int (f_0 - f_n)^+ \, d\mu \xrightarrow[n\to\infty]{} 0 \quad \text{.........} \quad (***)$$

from (*), (**) and (***)

$$\therefore \sup_{A \in \mathcal{A}} \left| \int_A f_n \, d\mu - \int_A f_0 \, d\mu \right| \xrightarrow[n\to\infty]{} 0$$

q.e.d.

Jon Norton.

Ex. 2.6.

Let $f_0, f_1, \dots, f_n \geq 0$, defined on $(\Omega, \mathcal{A}, \mu)$,

$f_n \xrightarrow[\mu]{} f_0$, $\int_\Omega f_n \, d\mu \xrightarrow[n\to\infty]{} \int_\Omega f_0 \, d\mu = 1$.

Show $\sup_{A \in \mathcal{A}} \left| \int_A f_n \, d\mu - \int_A f_0 \, d\mu \right| \xrightarrow[n\to\infty]{} 0$

Pf: $\forall A \in \mathcal{A}$ $\left| \int_A f_n \, d\mu - \int_A f_0 \, d\mu \right| = \left| \int_A (f_n - f_0) \, d\mu \right|$

$\leq \int_A |f_n - f_0| \, d\mu \leq \int_\Omega |f_n - f_0| \, d\mu$

Suffices to show $f_n \xrightarrow[1]{} f_0$.

Want $\{f_n\}_n \in \mathcal{L}_1$. $\exists N \ni: n \geq N \Rightarrow \int |f_n| \, d\mu = \int f_n \, d\mu < \infty$

So we start $\{f_n\}_n$ at $n = N$.

Now we can apply Vitali's theorem (p. 55) with $r = 1$, $X_n \equiv f_n$, $X \equiv f_0$. (Note we do <u>not</u> need $\mu(\Omega) < \infty$.) By Vitali,

$E|f_n| \xrightarrow[n\to\infty]{} E|f_0| \Rightarrow f_n \xrightarrow[1]{} f_0$. QED.

STA 5441 Hwk 1

1. (b).

To show $d_{TV}(P.Q)$ does not depend on the choice of μ.

Assume there are Lesbersgue measure μ_1. and μ_2, such that

$P.Q << \mu_1$ ~~wt~~ with R-N derivatives p_1 and q_1.

$P.Q << \mu_2$ with R-N derivatives p_2 and q_2.

$$d_{TV}(P,Q) \equiv \sup_{A\in\mathcal{A}} |P(A)-Q(A)| = \frac{1}{2}\int |p_1-q_1|\, d\mu_1$$

$$= \frac{1}{2}\int |p_2-q_2|\, d\mu_2 .$$

by definition. so not matter the choice of μ.

$d_{TV}(P.Q)$ all get the same value. #

3. $X_n \sim Bin(n.p_n)$. with $np_n \longrightarrow \lambda > 0$ as $n\to\infty$. P_n induced measu

$X_0 \sim Possion(\lambda)$ P_0 be its induced measure.

show that $d_{TV}(P_n.P_0) \to 0$ as $n\to\infty$

proof: P_n. P_0 are density with respect to μ. where μ is the counting measure defined on $\mathbb{Z}$.

example. $\mu((0,1))=0$, $\mu(\{1\})=1$, $\mu((\frac{1}{2},\frac{3}{2}))=1$.

so $\int_A P_n\, d\mu = \sum_{\substack{\#\text{integer}\\ \text{in } A}} P_n$

having this form. we can use the result from #2 (Scheffe's theore

we just need to show that

$P_n \xrightarrow{a.e} P_0$ as $n\to\infty$.

$$P_n(k) = \binom{n}{k} p_n^k (1-p_n)^{n-k}$$

$$= \frac{n!}{k!(n-k)!} p_n^k \cdot (1-p_n)^{n-k}$$

$$= \frac{n(n-1)\cdots(n-(k+1))}{k!} p_n^k (1-p_n)^{n-k}$$

$$= \frac{n^k (1)(1-\frac{1}{n})\cdots(1-\frac{k-1}{n}) p_n^k (1-p_n)^{n-k}}{k!}$$

$$= \frac{(np_n)^k}{k!} (1-p_n)^{n-k} \quad \underbrace{(1)(1-\tfrac{1}{n})\cdots(1-\tfrac{k-1}{n})}_{\to 1}$$

$$\frac{(np_n)^k}{k!} \to \frac{\lambda^k}{k!}$$

$$(1-p_n)^{n-k} = \frac{\left((1-p_n)^{\frac{1}{p_n}}\right)^{n p_n}}{(1-p_n)^k} \to (e^{-1})^{\lambda} = e^{-\lambda}$$

$$\to P_o(k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

Then use #2 a) result we get $d_{TV}(P_n, P_o) \to 0 \quad (n \to \infty)$.

Ex 1 $\quad P$ and Q are 2 prob meas on $(\Omega, \mathcal{A})$

$P << \mu$

$Q << \mu$

p and q are RN derivaties of P & Q wrt μ.

ie $\quad \dfrac{dP}{d\mu} = p$ and $\dfrac{dQ}{d\mu} = q$

a) $$d_{TV}(P,Q) \equiv \sup_{A \in \mathcal{A}} |P(A) - Q(A)| = \frac{1}{2} \int |p-q| \, d\mu$$
$$= \int (p-q)^+ d\mu.$$

Proof: $\quad$ Now any $X = X^+ - X^-$

so $\quad (p-q) = (p-q)^+ - (p-q)^-$

$$\int (p-q) d\mu = \int (p-q)^+ d\mu - \int (p-q)^- d\mu$$

Now $\quad \int p = 1 = \int q \Rightarrow \int (p-q)\, d\mu = 0$

$$\Rightarrow \int (p-q)^+ d\mu = \int (p-q)^- d\mu \qquad \left\{ \because X = X^+ - X^- \right.$$

Now $\quad |p-q| = (p-q)^+ + (p-q)^-$

$$\int |p-q| \, d\mu = \int (p-q)^+ d\mu + \int (p-q)^- d\mu$$
$$= 2 \int (p-q)^+ . \, d\mu$$

ie $$\frac{1}{2} \int |p-q| \, d\mu = \int (p-q)^+ d\mu$$

$$|P(A) - Q(A)| = \left| \int_A p\,d\mu - \int_A q\,d\mu \right|$$

Now to show $\sup\limits_{A \in \mathcal{A}} 2\,|P(A) - Q(A)| = \int |p-q|\,d\mu$

Now $2\,|P(A) - Q(A)| = |P(A) - Q(A)| + |P(A) - Q(A)|$

$$= |P(A) - Q(A)| + |1 - Q(A) - 1 + P(A)|$$

$$= |P(A) - Q(A)| + |P(A^c) + Q(A^c)| \qquad \{\because P(\Omega) = 1 = Q(\Omega)\}$$

$$= \left| \int_A p\,d\mu - \int_A q\,d\mu \right| + \left| \int_{A^c} p\,d\mu - \int_{A^c} q\,d\mu \right|$$

$$\leq \int_A |p-q|\,d\mu + \int_{A^c} |p-q|\,d\mu$$

$$= \int_\Omega |p-q|\,d\mu$$

So $\quad 2\,|P(A) - Q(A)| \leq \int |p-q|\,d\mu \quad$, for any $A \in \mathcal{A}$

$\Rightarrow \quad \sup\limits_{A \in \mathcal{A}} 2\,|P(A) - Q(A)| \leq \int |p-q|\,d\mu \quad$ — $(*)$

Now suppose $\sup\limits_{x \in A} \{x : x \in A\} \leq M$

If we can find an $x_0 \in A$ st $x_0 = M$ then $\sup\limits_{x \in A} \{x : x \in A\} = x_0 = M$

since $\sup\limits_{x \in A} \{x : x \in A\} \geq x_0 \Rightarrow \sup\limits_{x \in A} \geq M$ & we have $\sup\limits_{x \in A} \leq M$

We now show $\exists$ an A_0 st $|P(A_0) - Q(A_0)| = \frac{1}{2}\int |p-q| \cdot d\mu$

$$= \int (p-q)^+ d\mu$$

Let $A_0 \equiv \{ \omega : p-q \geq 0 \}$.

$$|P(A_0) - Q(A_0)| = \left| \int_{A_0} p-q \; d\mu \right|$$

$$= \left| \int_{\{p-q \geq 0\}} (p-q) \, d\mu \right|$$

$$= \left| \int (p-q)^+ d\mu \right| = \int (p-q)^+ \cdot d\mu.$$

$$\sup_{A \in \mathcal{A}} 2|P(A) - Q(A)| \geq 2|P(A_0) - Q(A_0)| = 2\int (p-q)^+ \cdot d\mu = \int |p-q| \, d\mu \quad \text{---} (**)$$

1 b) To show this is indep of the choice of μ

ie suppose we have another dominating meas μ_2 st

$P << \mu_2$ and $Q << \mu_2$ with RN derivatives

$p_2 = \dfrac{dP}{d\mu_2}$ and $q_2 = \dfrac{dQ}{d\mu_2}$

$$d_{VT}(P, Q) = \frac{1}{2}\int |p_2 - q_2| \cdot d\mu_2$$ and using 1(a) ....

$$= \sup_{A \in \mathcal{A}} |P(A) - Q(A)|$$ which is indep of the dominating meas.

$$= \frac{1}{2}\int |p-q| \, d\mu$$

$\underline{\text{Ex 2}}$ Very important tool to show weak convergence of r.v.

2) $f_0, f_1, \cdots f_n \cdots \geq 0$ and $\int_{\Omega} f_n \, d\mu = 1 \quad \forall\, n \geq 0$

Also given $f_n \xrightarrow{a.e} f_0$

Now $$\left| \int_A f_n \, d\mu - \int_A f_0 \, d\mu \right| \leq \int_A |f_n - f_0| \, d\mu$$

$$\leq \int_{\Omega} |f_n - f_0| \, d\mu \qquad \text{for any } A \in \mathcal{A}$$

$$\Rightarrow \sup_{A \in \mathcal{A}} \left| \int_A f_n \, d\mu - \int_A f_0 \, d\mu \right| \leq \int_{\Omega} |f_n - f_0| \, d\mu$$

So to show $\sup_{A \in \mathcal{A}} \left| \int_A f_n \, d\mu - \int_A f_0 \, d\mu \right| \longrightarrow 0$

it is enough to show $\int_{\Omega} |f_n - f_0| \cdot d\mu \longrightarrow 0$

ie to show $2 \int (f_n - f_0)^+ \cdot d\mu \longrightarrow 0$

OR $\int (f_n - f_0)^+ \, d\mu \longrightarrow 0$

Now we have $f_n \xrightarrow{a.e} f_0 \Rightarrow f_n - f_0 \xrightarrow{a.e} 0$

$\Rightarrow |f_n - f_0| \xrightarrow{a.e} 0$

Now $0 \leq (f_n - f_0)^+ \leq |f_n - f_0| \Rightarrow (f_n - f_0)^+ \xrightarrow{a.e} 0$

Now we will use the DCT to show $\int (f_n - f_0)^+ \longrightarrow 0$

Now $f_0 - f_n \leq f_0 \Rightarrow (f_0 - f_n)^+ \leq f_0$ $\left\{ \because \begin{array}{l} f_n \geq 0 \\ f_0 \geq 0 \end{array} \right.$

Now $f_0 \in \mathcal{L}_1$ since f_0 is a density

So we have

(a) $|(f_0 - f_n)^+| = (f_0 - f_n)^+ \leq f_0$ where $f_0 \in \mathcal{L}_1$

(b) $(f_n - f_0)^+ \xrightarrow{a.e} 0$

$\therefore$ by DCT we have $\int (f_n - f_0)^+ \cdot d\mu \longrightarrow 0$

QED

(b) Now we have $f_n \xrightarrow{\mu} f_0$ and $\int_\Omega f_n \, d\mu \longrightarrow \int_\Omega f \, d\mu$

$f_0, f_n \geq 0$ and $\int_\Omega f_0 \, d\mu = \infty$. $\left[\int_\Omega f_n \, d\mu \text{ need not } = 1 \right]$

<u>Approach</u> Vitali's Theorem : If $X_n \in \mathcal{L}_\mu$ and $X_n \xrightarrow{\mu} X$ and $\mu(\Omega) < \infty$

then $\left.\begin{array}{ll} a) & E|X_n|^2 \longrightarrow E|X|^2 \\ b) & X_n \xrightarrow{r} X \\ c) & \{X_n\} \text{ are u.i} \end{array}\right\}$ are equivalent

$\int f_n d\mu \longrightarrow \int f d\mu \Rightarrow$ given any $\varepsilon > 0$ $\exists N_\varepsilon$ st

$\left| \int f_n d\mu - \int f d\mu \right| < \varepsilon \quad \forall n \geq N_\varepsilon$

$\therefore \quad \int f_0 \, d\mu - \varepsilon \leq \int f_n \, d\mu \leq \int f_0 \, d\mu + \varepsilon$

Now $\int f_0 \, d\mu - \varepsilon$ and $\int f_0 \, d\mu + \varepsilon$ are both $< \infty$

$\Rightarrow \quad f_n \in \mathcal{L}_1$

We now have $f_n \xrightarrow{\mu} f_0$ and $f_n \in \mathcal{L}_1$

We will show $E|f_n| \longrightarrow E|f_0| \xRightarrow{\text{by Vitali's}} f_n \xrightarrow{\mathcal{L}_1} f$

Now $E|f_n| = \int |f_n| \, d\mu = \int f_n \, d\mu$

$E|f_0| = \int |f_0| \, d\mu = \int f_0 \, d\mu$

so $E|f_n| \longrightarrow E|f_0| \quad \{\because \int f_n \, d\mu \longrightarrow \int f_0 \, d\mu$

$\Rightarrow \quad E|f_n - f_0| \longrightarrow 0 \quad \{$By vitali's theorem

ie $\int |f_n - f_0| \longrightarrow 0$

Now $\left| \int_A f_n \, d\mu - \int_A f_0 \, d\mu \right| \leq \int_A |f_n - f_0| \, d\mu \leq \int |f_n - f_0| \, d\mu \longrightarrow 0$

So $\sup_{A \in \mathcal{A}} \left| \int_A f_n \, d\mu - \int_A f_0 \, d\mu \right| \longrightarrow 0$

Ex 3 $\quad X_n \sim Bin(n, p_n)$ with $np_n \longrightarrow \lambda \quad ; (\lambda > 0)$

P_n is the induced dist^n of X_n on $\mathbb{R}$ (ie $P_n \circ X_n^{-1}$)

$X_0 \sim Pois(\lambda)$

P_0 is induced dist^n of X_0 on $\mathbb{R}$. (ie $P_0 \circ X_0^{-1}$)

To show $d_{TV}(P_n, P_0) = \sup\limits_{A \in \mathcal{A}} \left| P_n(A) - P_0(A) \right| \longrightarrow 0$ ✓

ie to show

$$\times \quad \frac{1}{2} \int \left| p_n - p_0 \right| d\mu \longrightarrow 0$$

We 1st need to find a measure st $P_n << \mu$ and

$P_0 << \mu$

and define the RN derivatives wrt μ

We can use Ex 2 and show $p_n \xrightarrow{a.e} p_0$ (Sheffé theorem)

then $\Rightarrow \quad \sup\limits_{A \in \mathcal{A}} \left| \int_A p_n - \int_A p_0 \right| \longrightarrow 0$

1) Select $\mu \equiv$ counting meas on $\mathbb{Z}$

2) Select $p_n \equiv Bin(n, p_n)$ density function

$p_0 \equiv Poiss(\lambda)$ density function

$$M_{X_n}(t) = [p_n e^t + (1-p_n)]^n$$

$$M_{X_0}(t) = e^{\lambda(e^t - 1)}$$

Now $np_n \longrightarrow \lambda$

$$M_{X_n}(t) = [p_n e^t + 1 - p_n]^n$$

$$= [p_n(1+e^t) + 1]^n$$

$$= \left[\frac{np_n}{n}\left(1+e^t\right) + 1\right]^n$$

$$\longrightarrow e^{\lambda(e^t - 1)}$$

$\left\{ \begin{array}{l} \text{Fact} \\ \left(1+\frac{x}{n}\right)^n \longrightarrow e^x \end{array} \right.$

Must include the fact that $f_n << \mu$, $f_0 << \mu$

and the fact that the $\int f_n$ & $\int f_0$ makes sense and is the RN-derivative of P_n & P_0 wrt μ on $\mathbb{Z}$

Note : $d_{VT}(P_n, P_0) = \frac{1}{2} \sum_{k=0}^{\infty} [P(X_n = k) - P(X_0 = k)] = 0$

we only needed $[P(X_n = k) - P(X_0 = k)] \longrightarrow 0$

But in general $\sum \frac{1}{n} = \infty$ even when $\frac{1}{n} \longrightarrow 0$

Homework 2 – STA 5447, Spring 2006.

Assigned Th, Jan the 26^{th}
Due Tue, Feb the 7^{th}.

Useful for mixture densities. the densities are well defined.

(1). Exercise 4.2.1 page 67 PfS.

If Normal is abs cont wrt cauchy & RN theorem says

(2) Exercise 4.2.2 page 67 PfS.

(3) Show the following: (see also Prop. 7.4.2 page 117 and Ex. 7.4.2).

a) If $X \geq 0$ has df F then:

$$EX = \int_0^\infty (1 - F(x))\,dx.$$

b) If $X \geq 0$ has df F then:

$$EX^r = r\int_0^\infty x^{r-1}(1 - F(x))\,dx, \text{ for } r > 1.$$

c) If $E|X| < \infty$ then

$$EX = -\int_{-\infty}^0 F(x)\,dx + \int_0^\infty (1 - F(x))\,dx.$$

4. Show that if $X \geq 0$ is integer valued

$$EX = \sum_{K=1}^{\infty} P(X \geq K).$$

$$EX^2 = \sum_{K=0}^{\infty} (2K+1) P(X > K).$$

5. Let $A_1 = [0,1]$ and $A_2 = (1, \infty)$, both equipped with the Borel sets and Lebesgue measure.

Let $f(x,y) = e^{-xy} - 2e^{-2xy}$.

Show that:

Show bdd (lim at $x=0$ & $x=1$) & continuous
∴ on $(0,1)$ it is integrable

a) $$\int_0^1 \left(\int_1^{\infty} f(x,y)\,dy \right) dx = \int_0^1 \boxed{\frac{1}{x}\left(e^{-x} - e^{-2x}\right)}\, dx \text{ exists}$$
and is $\boxed{>0}$.

Show bdd above by $(e^{-2y} - e^{-y})$ which is integrable (evaluate it)

b) $$\int_1^{\infty} \left(\int_0^1 f(x,y)\,dx \right) dy = \int_1^{\infty} \boxed{\frac{1}{y}\left(e^{-2y} - e^{-y}\right)}\, dy \text{ exists}$$
and is $\boxed{<0}$.

Does this contradict Fubini or Tonelli theorems?

Hwk presentation:
1. Ex. 1 and 2
2. Ex. 3.
3. Ex 4 and 5.

Show $|f(x,y)| \geq$ const

$\int_0^1 \int_0^{\infty}$ const $= \infty$ so conditions are violated.

1]. μ and ν are σ-finite meas on $(\Omega, \mathcal{A})$
Φ and Ψ are σ-finite signed meas on $(\Omega, \mathcal{A})$

(a) $\Phi << \mu$ and $\Psi << \mu$. Show $\frac{d(\Psi+\Phi)}{d\mu} = \frac{d\Phi}{d\mu} + \frac{d\Psi}{d\mu}$ a.e μ

(b) $\Phi << \mu$ and $\mu << \nu$. Show $\frac{d\Phi}{d\nu} = \frac{d\Phi}{d\mu} \cdot \frac{d\mu}{d\nu}$ a.e ν

Solⁿ: 1st we show $(\Phi+\Psi) << \mu$

$\Phi << \mu \Rightarrow \frac{d\Phi}{d\mu}$ is the RN derivative & $\Phi(A) = \int_A \frac{d\Phi}{d\mu} d\mu.$

$\Psi << \mu \Rightarrow \frac{d\Psi}{d\mu}$ is the RN derivative & $\Psi(A) = \int_A \frac{d\Psi}{d\mu} d\mu$

Now $\mu(A) = 0 \Rightarrow \Phi(A) = 0$ and $\Psi(A) = 0$

$\Rightarrow \Phi(A) + \Psi(A) = 0$

$\Rightarrow (\Phi+\Psi)(A) = 0$

$\therefore \Phi+\Psi << \mu \Rightarrow (\Phi+\Psi)(A) = \int_A \frac{d(\Phi+\Psi)}{d\mu} d\mu$ by RN theorem

$$\int_A \frac{d(\Phi+\Psi)}{d\mu} d\mu = (\Phi+\Psi)(A)$$

$$= \Phi(A) + \Psi(A)$$

$$= \int_A \frac{d\Phi}{d\mu} d\mu + \int_A \frac{d\Psi}{d\mu} d\mu$$ and this holds for any $A \in \mathcal{A}$

$$\Rightarrow \int_A \frac{d(\Phi+\Psi)}{d\mu} d\mu = \int_A \left(\frac{d\Phi}{d\mu} + \frac{d\Psi}{d\mu} \right) d\mu \Rightarrow \frac{d(\Psi+\Phi)}{d\mu} = \frac{d\Phi}{d\mu} + \frac{d\Psi}{d\mu}$$

b) $\phi << \mu \Rightarrow \dfrac{d\phi}{d\mu}$ is the RN derivative $\phi(A) = \int_A \dfrac{d\phi}{d\mu} \cdot d\mu$

$\mu << \nu \Rightarrow \dfrac{d\mu}{d\nu}$ is the RN derivative $\mu(A) = \int_A \dfrac{d\mu}{d\nu} \cdot d\nu$

Now to show $\Phi << \nu$.

if $\nu(A) = 0 \Rightarrow \mu(A) = 0$

$\Rightarrow \Phi(A) = 0$

$\therefore \dfrac{d\phi}{d\nu}$ is the RN derivative $\Phi(A) = \int_A \dfrac{d\phi}{d\nu} \cdot d\nu$

Now $\int_A \dfrac{d\phi}{d\nu} d\nu = \Phi(A) = \int_A \dfrac{d\phi}{d\mu} \cdot d\mu$

$= \int_A \dfrac{d\phi}{d\mu} \dfrac{d\mu}{d\nu} \cdot d\nu$ $\left\{ \begin{array}{l} \text{by change of variable} \\ \because \mu << \nu \ \& \ \int_A \frac{d\phi}{d\mu} \cdot d\mu \\ \text{is well defined} \end{array} \right.$

$\Rightarrow \int_A \left(\dfrac{d\phi}{d\nu}\right) d\nu = \int_A \dfrac{d\phi}{d\mu} \cdot \dfrac{d\mu}{d\nu} \cdot d\nu$ for any $A \in \mathcal{A}$

$\Rightarrow \dfrac{d\phi}{d\nu} = \dfrac{d\phi}{d\mu} \dfrac{d\mu}{d\nu}$ a.e ν $\{$by uniqueness of the RN theorem

2] . $P_{\mu,\sigma^2} \simeq N(\mu, \sigma^2)$ distrubution

P is cauchy dustribution

Show $P_{\mu,1} << P_{0,1}$ and compute $\dfrac{dP_{\mu,1}}{dP_{0,1}}$

Sol :

$$P_{\mu,\sigma^2}(A) = \int_A \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{\frac{-1}{2\sigma^2}(x-\mu)^2\right\} \cdot d\lambda(x)$$

$$\equiv \int_A f_{\mu,\sigma^2}(x)\, dx$$

$$P(A) = \int_A \frac{1}{2\pi} \frac{1}{(1+x^2)} dx$$

Now $P_{\mu,1}(A) = \int_A f_{\mu,1}(x)\, dx = \int_A \frac{f_{\mu,1}}{f_{0,1}} f_{0,1}\, dx$

$$= \int_A \frac{f_{\mu,1}}{f_{0,1}} \frac{dP_{0,1}}{d\lambda} d\lambda(x)$$

$$= \int_A \frac{f_{\mu,1}}{f_{0,1}} dP_{0,1}$$ { by change of variable theorem $\because P_{0,1} << \lambda$

If $P_{0,1}(A) = 0 \Rightarrow P_{\mu,1}(A) = 0$ { by abs continuity of the integral

$\therefore$ $P_{\mu,1} << P_{0,1}$

Now $\dfrac{dP_{\mu,1}}{dP_{0,1}} = \dfrac{dP_{\mu,1}}{d\lambda}\dfrac{d\lambda}{dP_{0,1}} = \dfrac{dP_{\mu,1}}{d\lambda}\dfrac{1}{\left(\frac{dP_{0,1}}{d\lambda}\right)} = \dfrac{f_{\mu,1}}{f_{0,1}}$

(b) Show $P_{0,\sigma^2} << P_{0,1}$ & compute $\dfrac{dP_{0,\sigma^2}}{dP_{0,1}}$

$$P_{0,\sigma^2}(A) = \int_A \frac{dP_{0,\sigma^2}}{d\lambda}\, d\lambda = \int_A f_{0,\sigma^2}(x)\, dx$$

$$= \int_A \left(\frac{f_{0,\sigma^2}}{f_{0,1}}\right) f_{0,1}\, d\lambda(x)$$

$$= \int_A \left(\frac{f_{0,\sigma^2}}{f_{0,1}}\right) \frac{dP_{0,1}}{d\lambda}\, d\lambda$$

$$= \int_A \left(\frac{f_{0,\sigma^2}}{f_{0,1}}\right) dP_{0,1}$$ { change of variable

$\therefore$ If $P_{0,1}(A) = 0 \Rightarrow P_{0,\sigma^2}(A) = 0$ { abs continuity of integral

$$\frac{dP_{0,\sigma^2}}{dP_{0,1}} = \frac{dP_{0,\sigma^2}}{d\lambda}\frac{d\lambda}{dP_{0,1}}$$ [can show $\lambda << P_{0,1}$

$$= \left(\frac{dP_{0,\sigma^2}}{d\lambda}\right)\frac{1}{\left(\frac{dP_{0,1}}{d\lambda}\right)}$$

$$= \frac{f_{0,\sigma^2}}{f_{0,1}}$$

(c) Show $P << P_{0,1}$

$$P(A) = \int_A \frac{1}{2\pi} \frac{1}{(1+x^2)} dx \equiv \int_A g(x)\, dx = \int_A \frac{g(x)}{f_{0,1}} f_{0,1}\, d\lambda(x)$$

$$= \int_A \frac{g(x)}{f_{0,1}} \frac{dP_{0,1}}{d\lambda} d\lambda$$

$$= \int_A \frac{g(x)}{f_{0,1}(x)} dP_{0,1}(x) \quad \{ \text{change of variables}$$

If $P_{0,1}(A) = 0 \Rightarrow P(A) = 0$ $\quad$ {by abs continuity of the integral

$$\frac{dP}{dP_{01}} = \frac{dP}{d\lambda} \frac{d\lambda}{dP_{0,1}} = \left(\frac{dP}{d\lambda}\right) \frac{1}{\left(\frac{dP_{01}}{d\lambda}\right)} = \frac{g}{f_{0,1}}$$

To show $\lambda << P_{0,1}$

$$P_{0,1}(A) = 0 \Rightarrow 0 = \int_A \frac{dP_{0,1}}{d\lambda} d\lambda \Rightarrow \int_A f_{0,1}\, d\lambda = 0 \quad \text{with } f_{0,1} > 0 \text{ a.e.}$$

$$\Rightarrow \lambda(A) = 0$$

To show $\mu << \lambda$ and $\lambda << \mu$ then $\frac{d\mu}{d\lambda}\frac{d\lambda}{d\mu} = 1$

$$\int_A \left(\frac{d\mu}{d\lambda} \frac{d\lambda}{d\mu}\right) d\mu \overset{①}{=} \int_A \frac{d\mu}{d\lambda} d\lambda \overset{②}{=} \mu(A) \qquad \begin{cases} ① \text{ change of variable} \\ ② \text{ RN theorem} \end{cases}$$

$$= \int_A d\mu$$

$$\Rightarrow \frac{d\mu}{d\lambda}\frac{d\lambda}{d\mu} = 1 \quad \text{a.e } \mu$$

|||ly $\displaystyle \int_A \frac{d\mu}{d\lambda}\frac{d\lambda}{d\mu} d\lambda = \int_A \frac{d\lambda}{d\mu} d\mu = \lambda(A) = \int_A d\lambda \Rightarrow \frac{d\mu}{d\lambda}\frac{d\lambda}{d\mu} = 1 \text{ a.e } \lambda$

3] a) $X \geq 0$ with df F then $E(X) = \int_0^\infty (1-F(x))\,dx$

$$E(X) = \int_\Omega X(\omega)\,dP(\omega)$$

$$= \int_\Omega \int_0^{X(\omega)} d\lambda\, dP(\omega)$$

$$= \int_\Omega \int_0^\infty I_{[0<t<X(\omega)]}\, d\lambda(t)\, dP(\omega)$$ $\left\{\begin{array}{l}\text{defined on } (\Omega \times \mathbb{R}, \mathcal{A} \times \mathcal{B}) \\ \text{with } \Phi = \lambda \times P\end{array}\right.$

$$= \int_0^\infty \int_\Omega I_{[0<t<X(\omega)]}\, dP(\omega)\, d\lambda(t)$$ $\left\{\begin{array}{l}\text{by Tonelli since} \\ I_{[\ ]} \geq 0\end{array}\right.$

$$= \int_0^\infty P(X(\omega) > t)\, d\lambda(t)$$

$$= \int_0^\infty [1-F(t)]\,dt$$

(b) $X \geq 0$ with df F then $E(X^r) = r\int_0^\infty x^{r-1}(1-F(x))\,dx$, $r > 1$

$$E(X^r) = \int_\Omega (X(\omega))^r\, dP(\omega) = \int_\Omega \int_0^{X(\omega)} r t^{r-1}\, d\lambda(t)\, dP(\omega)$$

$$= \int_\Omega \int_0^\infty r\, I_{[0<t<X(\omega)]}\, t^{r-1}\, d\lambda(t)\, dP(\omega)$$

$$= \int_0^\infty \int_\Omega r\, I_{[0<t<X(\omega)]}\, t^{r-1}\, dP(\omega)\, d\lambda(t)$$ $\left\{\begin{array}{l}\text{by Tonelli's} \\ t^{r-1} I_{[0<t<X(\omega)]} \geq 0\end{array}\right.$

$$= \int_0^\infty r\, P(X(\omega) > t)\, t^{r-1}\, d\lambda(t) = \int_0^\infty r t^{r-1}(1-F(t))\,dt$$

c) If $E|X| < \infty$ then $E(X) = \int_0^\infty [1-F(t)]\,dt - \int_{-\infty}^0 F(t)\,dt$

$X = X^+ - X^-$

$\therefore\ E(X) = E(X^+) - E(X^-) \quad < \infty \Rightarrow E(X^+) < \infty$ and $E(X^-) < \infty$

Now $X^+ \geq 0 \Rightarrow E(X^+) = \int_\Omega X^+ dP = \int_\Omega \int_0^{X^+} d\lambda\, dP = \int_\Omega \int_0^\infty I_{[0<t<X^+]}\, d\lambda\, dP$

$$= \int_0^\infty \int_\Omega I_{[0<t<X^+]}\, dP\, d\lambda(t) \qquad \{\because \text{Tonelli}$$

$$= \int_0^\infty P(X^+ > t)\, d\lambda(t)$$

$$= \int_0^\infty [1-F(t)]\, dt \qquad \{\because \{\omega | X^+(\omega) > t\} = \{\omega | X(\omega) > t\}$$

Now $X^- \geq 0 \Rightarrow E(X^-) = \int_\Omega X^- dP = \int_\Omega \int_{-X^-(\omega)}^0 d\lambda(t)\, dP(\omega)$

$$= \int_\Omega \int_{-\infty}^0 I_{[-X^- \leq t < 0]}\, d\lambda(t)\, dP(\omega)$$

$$= \int_{-\infty}^0 \int_\Omega I_{[-X^- \leq t]}\, dP(\omega)\, d\lambda(t) \qquad \{\text{Tonelli}$$

$$= \int_{-\infty}^0 P(-X^- \leq t)\, dt$$

$$= \int_{-\infty}^0 P(X \leq t)\, dt \qquad \{\because \{\omega | X(\omega) \leq t\} = \{\omega | -X^-(\omega) < t\}$$

To show $\{\omega \mid X(\omega) \le t\} = \{\omega \mid -X^-(\omega) \le t\}$

Let $\omega_0 \in \{\omega \mid X(\omega) \le t\} \Rightarrow X(\omega_0) \le t \Rightarrow X^+(\omega_0) - X^-(\omega_0) \le t$

$\Rightarrow -X^-(\omega_0) \le t - X^+(\omega_0) \le t$

Let $\omega_0 \in \{\omega \mid -X^-(\omega) \le t\} \Rightarrow -X^-(\omega_0) \le t$

Now $-X^-(\omega_0) = X(\omega_0)$ if $X(\omega_0) \le 0$
$= 0$ ow

Also $t < 0 \Rightarrow -X^-(\omega_0) < 0$
$\rightarrow -X^-(\omega_0) = X(\omega_0) \le t$

To show $\{\omega \mid X^+(\omega) > t\} = \{\omega \mid X(\omega) > t\}$; $t > 0$

$X^+(\omega) = X(\omega)$ if $X(\omega) \ge 0$
0 ow

If $X(\omega_0) > t > 0 \Rightarrow X^+(\omega_0) = X(\omega_0)$
If $X^+(\omega_0) > t > 0 \Rightarrow X^+(\omega_0) = X(\omega_0)$

4] If $X \ge 0$ integer value. Show $E(X) = \sum_{k=1}^{\infty} P(X \ge k)$

$$E(X^2) = \sum_{k=0}^{\infty} (2k+1) P(X > k)$$

$$E(X) = \int_0^{\infty} (1 - F(x))\, dx = \sum_{k=0}^{\infty} \int_{[k,k+1)} (1-F(x))\, dx$$

Now on $[k, k+1)$, $F(x)$ takes const value $= F(k)$

$$E(X) = \sum_{k=0}^{\infty} (1-F(k)) \int_{[k,k+1)} dx = \sum_{k=0}^{\infty} (1-F(k)) = \sum_{k=0}^{\infty} P(X > k) = \sum_{k=0}^{\infty} P(X \ge k+1)$$

$$= \sum_{k=1}^{\infty} P(X \ge k)$$

$$E(X^2) = \int_0^\infty 2t\,(1-F(t))\,dt$$

$$= \sum_{k=0}^{\infty} \int_{[k,k+1)} 2t\,(1-F(t))\,dt \quad = \sum_{k=0}^{\infty} (1-F(k)) \int_{[k,k+1)} 2t\,dt$$

$$= \sum_{k=0}^{\infty} (1-F(k)) \left(t^2 \Big|_k^{k+1} \right) \quad = \sum_{k=0}^{\infty} (1-F(k))\,((k+1)^2 - k^2)$$

$$= \sum_{k=0}^{\infty} (2k+1)\,P(X>k)$$

5] $A_1 = [0,1]$ $\qquad A_2 = (1,\infty)$

$f(x,y) = e^{-xy} - 2e^{-2xy}$

Show a) $\int_0^1 \left(\int_1^\infty f(x,y)\,dy \right) dx = \int_0^1 \frac{1}{x}\left(e^{-x} - e^{-2x}\right) dx$ exists & >0

b) $\int_1^\infty \left(\int_0^1 f(x,y)\,dx \right) dy = \int_1^\infty \frac{1}{y}\left(e^{-2y} - e^{-y}\right) dy$ exists & <0

c) Does it contradict Tonelli's theorem

a) $$\int_0^1 \int_1^\infty f(x,y)\,dy\,dx = \int_0^1 \left(\int_1^\infty e^{-xy} - 2e^{-2xy}\,dy \right) dx$$

$$= \int_0^1 \frac{1}{x}\left(e^{-x} - e^{-2x}\right) dx$$

Now $\frac{1}{x}\left(e^{-x} - e^{-2x}\right) > 0$ on $[0,1]$ $\Rightarrow \int_0^1 \frac{1}{x}\left(e^{-x} - e^{-2x}\right) dx > 0$

$$\lim_{x \to 0} \frac{e^{-x} - e^{-2x}}{x} = \lim_{x \to 0} -e^{-x} + 2e^{-x} = 1 < \infty$$

$$\lim_{x \to 1} \frac{e^{-x} - e^{-2x}}{x} = e^{-1} - e^{-2} < \infty$$

$\frac{1}{x}(e^{-x} - e^{-2x})$ is bdd on $[0,1]$ and is continuous on $[0,1]$ so it is also integrable

b) $$\int_1^\infty \left(\int_0^1 e^{-xy} - 2e^{-2xy} \, dx \right) dy = \int_1^\infty \frac{1}{y} (e^{-2y} - e^{-y}) \, dy$$

Now $\frac{1}{y}(e^{-2y} - e^{-y}) < 0 \Rightarrow \int_1^\infty \frac{1}{y}(e^{-2y} - e^{-y}) \, dy < 0$

Also $\frac{1}{y}(e^{-2y} - e^{-y}) < e^{-2y} - e^{-y}$ for $y \in (1, \infty)$

$$\int_1^\infty \frac{1}{y}(e^{-2y} - e^{-y}) \, dy < \int_1^\infty (e^{-2y} - e^{-y}) \, dy$$

$$= \left(\frac{e^{-2y}}{-2} - \frac{e^{-y}}{-1} \Big|_1^\infty \right)$$

$$= \left(\frac{e^{-2}}{2} - \frac{e^{-1}}{1} \right)$$

$$< \infty$$

c) Tonelli's theorem is not contradicted since the conditions req are all violated.

(i) $e^{-xy}-2e^{-2xy} < 0$ when $e^{-xy} < 2e^{-2xy}$ ie $-xy < \log 2 - 2xy$

ie $xy < \log 2$

$e^{-xy}-2e^{-2xy} > 0$ when $xy > \log 2$

(ii) $$\int_1^\infty \int_0^1 | e^{-xy}-2e^{-2xy} | \, dx \, dy$$

$$= \int_1^\infty \int_0^{\log 2/y} (-e^{-xy}+2e^{-2xy})\,dx\,dy + \int_1^\infty \int_{\frac{\log 2}{y}}^1 (e^{-xy}-2e^{-2xy})\,dx\,dy$$

$$= \int_1^\infty \left(\frac{1}{4y}\right) dy + \int_1^\infty \left[\frac{1}{y}(e^{-2y}-e^{-y}) + \frac{1}{4y}\right] dy$$

$$= \int_1^\infty \left(\frac{1}{2y}\right) dy + \underbrace{\int_1^\infty \frac{1}{y}(e^{-2y}-e^{-y})\,dy}_{<\infty}$$

$$= \frac{1}{2}\log y \Big|_1^\infty + (<\infty)$$

$$= \infty$$

(iii) $$\int_0^1 \int_1^\infty |e^{-xy}-2e^{-2xy}| \, dy \, dx$$

$$= \int_0^{\log 2} \int_1^{\log 2/x} -f(x,y)\,dy\,dx + \int_0^{\log 2} \int_{\frac{\log 2}{x}}^\infty f(x,y)\,dy\,dx + \int_{\log 2}^1 \int_1^\infty f(x,y)\,dy\,dx$$

$$= \int_0^{\log 2} \frac{1}{x}\left(e^{-2x}-e^{-x}+\frac{1}{4}\right) dx + \int_0^{\log 2} \frac{1}{4x}\,dx + \int_{\log 2}^1 \frac{1}{x}(e^{-x}-e^{-2x})\,dx = -\infty$$

Now $\int_0^{\log 2} \frac{1}{2x}\,dx = \frac{1}{2}\log x \Big|_0^{\log 2} = -\infty$

QED

Exercise 2.1 : Let μ & ν be σ-finite measures on $(\Omega, \mathcal{A})$. Let ϕ & ψ be σ-finite

red measures on $(\Omega, \mathcal{A})$. Then

$$\frac{d(\phi+\psi)}{d\mu} = \frac{d\phi}{d\mu} + \frac{d\psi}{d\mu} \quad \text{a.e. } \mu \text{ if } \phi \ll \mu \text{ and } \psi \ll \mu.$$

if we can show $\phi+\psi \ll \mu$, Then using the R-N Theorem we can write $(\phi+\psi)(A) = \int_A \frac{d\phi+\psi}{d\mu} d$ $\forall A \in \mathcal{A}$

To show $\phi+\psi \ll \mu$:

we have to show that if $\mu(A) = 0 \Rightarrow (\phi+\psi)(A) = 0 \quad \forall A \in \mathcal{A}$.

Proof: if $\mu(A) = 0$; $\phi(A) = 0$ & $\psi(A) = 0$ as $\phi \ll \mu$ & $\psi \ll \mu$.

Now $(\phi+\psi)(A) = \phi(A) + \psi(A) = 0 + 0 = 0$.

So $\mu(A) = 0 \Rightarrow (\phi+\psi)(A) = 0$ or $\phi+\psi \ll \mu$. $\square$

Now using R-N Theorem we can write

$$(\phi+\psi)(A) = \int_A \frac{d(\phi+\psi)}{d\mu} \, d\mu$$

as $\phi+\psi \ll \mu$

$\phi+\psi$ is a finite measure on $(\Omega, \mathcal{A})$.

Now we want to ~~show~~ use if

$$\int_A X(\omega)\, d\mu(\omega) = \int_A Y(\omega)\, d\mu(\omega) \quad \forall A \in \mathcal{A} \quad \text{then } X = Y$$

Now

$$\underset{\text{LHS}}{\phi(A) + \psi(A)} = \underset{\text{RHS}}{(\phi+\psi)(A)}$$

$$\int_A \frac{d\phi}{d\mu} d\mu + \int_A \frac{d\psi}{d\mu} d\mu = \underbrace{\int_A \left(\frac{d\phi}{d\mu} + \frac{d\psi}{d\mu}\right) d\mu}_{\text{L.H.S.}} \qquad \underbrace{\int_A \frac{d(\phi+\psi)}{d\mu} d\mu}_{\text{RHS}} \overset{\text{By RN}}{=} (\phi+\psi)A$$

RN: $\exists$ an a.e. μ unique ; μ integrable function $Z_0 \geq 0$. Show case $Z_0 = \frac{d\phi}{d\mu} + \frac{d\psi}{d\mu}$

And By Step 1 : $\frac{d\phi}{d\mu} + \frac{d\psi}{d\mu} = \frac{d(\phi+\mu)}{d\mu}$

Exercise 2.1

7\. Show $\frac{d\phi}{d\nu} = \frac{d\phi}{d\mu} \cdot \frac{d\mu}{d\nu}$ a.e. ν. if $\phi << \mu$; $\mu << \nu$

→ Step 1 : Show $\phi << \nu$

That's if $\nu(A) = 0 \Rightarrow \phi(A) = 0$ $\forall A \in \mathcal{A}$.

$\nu(A) = 0 \Rightarrow \mu(A) = 0$ as $\mu << \nu$

$\mu(A) = 0 \Rightarrow \phi(A) = 0$ as $\phi << \mu$. $\forall A \in \mathcal{A}$.

So $\nu(A) = 0$ implies $\phi(A) = 0$ So $\phi << \nu$

Now show $\frac{d\phi}{d\nu} = \frac{d\phi}{d\mu} \cdot \frac{d\mu}{d\nu}$ a.e. ν

$*$ $\phi(A) = \int_A \frac{d\phi}{d\mu} d\mu$ as $\phi << \mu$

$**$ $\phi(A) = \int_A \frac{d\phi}{d\nu} d\nu$ as $\phi << \nu$

μ, ν = σ-finite measures on $(\Omega, \mathcal{A})$. // helps us write the RN derivative.

Also Note $\frac{d\phi}{d\mu}$ & $\frac{d\phi}{d\nu}$ are well defined as they are RN derivative of ϕ with respect to μ & ν, respectively.

So $\phi(A) \overset{\text{By } *}{=} \int_A \frac{d\phi}{d\mu} d\mu = \int_A \frac{d\phi}{d\mu} \frac{d\mu}{d\nu} d\nu \quad \int_A \frac{d\phi}{d\nu} d\nu \overset{\text{By } **}{=} \phi(A)$.

By change of variable theorem (Theorem 2.2. page 66)

So $\int_A \frac{d\phi}{d\mu} \frac{d\mu}{d\nu} d\nu = \int_A \frac{d\phi}{d\nu} . d\nu$.

$\Rightarrow \frac{d\phi}{d\mu} \cdot \frac{d\mu}{d\nu} = \frac{d\phi}{d\nu}$.

By uniqueness of the RN derivative

f_{μ,σ^2} = density of $N(\mu,\sigma^2)$

Exercise 2.2

(a) Show that $P_{\mu,1} << P_{0,1}$ and compute $dP_{\mu,1}/dP_{0,1}$; $P_{\mu,\sigma^2} = N(\mu,\sigma^2)$; P = Cauchy.

want to show if $P_{0,1}(A)=0 \Rightarrow P_{\mu,1}(A)=0$

$P_{\mu,1}(A) = \int_A \frac{dP_{\mu,1}}{d\lambda} \cdot d\lambda$ $\quad \forall A \in \mathcal{A}$; as $\boxed{P_{\mu,1} << \lambda}$ (last page), and By RN Theorem.

$= \int_A f_{\mu,1}\, d\lambda = \int_A \boxed{\frac{f_{\mu,1}}{f_{0,1}} f_{0,1}\, d\lambda}$ (multiplying $f_{\mu,1}$ by $1 = \frac{f_{0,1}}{f_{0,1}}$ and $f_{0,1} > 0$)

$= \boxed{\int_A \frac{f_{\mu,1}}{f_{0,1}} \boxed{\frac{dP_{0,1}}{d\lambda}} \cdot d\lambda} = \boxed{\int_A \boxed{\frac{f_{\mu,1}}{f_{0,1}}}\, dP_{0,1}}$. Now since $\frac{f_{\mu,1}}{f_{0,1}} > 0$ $P_{0,1}(A)$ implies $P_{\mu,1}(A)=0$ By Absolute continuity of the Integral.

rewriting $f_{0,1}$

So $P_{\mu,1} << P_{0,1}$ $\square$

ALTERNATIVELY: John had suggested in class since we can show $P_{\mu,1} << \lambda$ and $\lambda << P_{0,1}$; then this implies $P_{\mu,1} << P_{0,1}$ $\square$.

compute : $\frac{dP_{\mu,1}}{dP_{0,1}} = ?$

$$\frac{dP_{\mu,1}}{dP_{0,1}} = \frac{dP_{\mu,1}}{d\lambda} \cdot \frac{d\lambda}{dP_{0,1}}$$

Using Exercise 4.21 and as ($P_{\mu,1} << \lambda$ & $\lambda << P_{0,1}$) shown on the last page.

$$\text{"}\frac{dP_{\mu,1}}{dP_{0,1}} = \frac{dP_{\mu,1}}{d\lambda} \frac{d\lambda}{dP_{0,1}} = \frac{dP_{\mu,1}}{d\lambda} \cdot \frac{1}{\frac{dP_{0,1}}{d\lambda}} = \boxed{\frac{f_{\mu,1}}{f_{0,1}}}$$

Justification of This Step:

If $\mu << \lambda$ & $\lambda << \mu$ then $\frac{d\mu}{d\lambda}\frac{d\lambda}{d\mu} = 1$.

Proof:

$\int \frac{d\mu}{d\lambda} \cdot \frac{d\lambda}{d\mu}\, d\mu = \int_A 1\, d\mu$? $\forall A \in \mathcal{A}$.

$\int_A \frac{d\mu}{d\lambda} \cdot \frac{d\lambda}{d\mu}\, d\mu = \int_A \frac{d\mu}{d\lambda}\, d\lambda = \mu(A) = \int_A 1\, d\mu$.

(Th: 2.2. Change of measure). By Definition

Similarly.

$\int_A \frac{d\mu}{d\lambda} \cdot \frac{d\lambda}{d\mu}\, d\lambda = \int_A \frac{d\lambda}{d\mu}\, d\mu = \lambda(A) = \int_A 1\, d\lambda$ $\square$.

By Definition

Exercise 2.2

b) Show $P_{0,\sigma^2} << P_{0,1}$ and compute $\frac{dP_{0,\sigma^2}}{dP_{0,1}}$

$$P_{0,\sigma^2} = \int_A \frac{dP_{0,\sigma^2}}{d\lambda} \cdot d\lambda = \int_A f_{0,\sigma^2}\, d\lambda = \int_A \frac{f_{0,\sigma^2}}{f_{0,1}} f_{0,1}\, d\lambda$$ (multiplying f_{0,σ^2} by 1 as $f_{0,1} > 0$).

($P_{0,\sigma^2} << \lambda$ last page).

$$= \int_A \frac{f_{0,\sigma^2}}{f_{0,1}} \frac{dP_{0,1}}{d\lambda} \cdot d\lambda = \int_A \frac{f_{0,\sigma^2}}{f_{0,1}}\, dP_{0,1}$$

Now since $\frac{f_{0,\sigma^2}}{f_{0,1}} > 0$ If $P_{0,1}(A) = 0$ then $P_{0,\sigma^2}(A) = 0$. By Absolute Continuity of the Integral.

compute

$$\frac{dP_{0,\sigma^2}}{dP_{0,1}} = \frac{dP_{0,\sigma^2}}{d\lambda} \cdot \frac{d\lambda}{dP_{0,1}} = \frac{dP_{0,\sigma^2}}{d\lambda} \cdot \frac{1}{\frac{dP_{0,1}}{d\lambda}} = \boxed{\frac{f_{0,\sigma^2}}{f_{0,1}}}$$

Exercise 2.2 (c)

compute

$$\frac{dP}{dP_{0,1}} = \frac{dP}{d\lambda} \cdot \frac{d\lambda}{dP_{0,1}} = \frac{dP}{d\lambda} \cdot \frac{1}{\frac{dP_{0,1}}{d\lambda}} = \boxed{\frac{f_{cauchy}}{f_{0,1}}}$$

where f_{cauchy} = density of the Cauchy Distribution

Similarly:

$$\frac{dP_{0,1}}{dP} = \frac{dP_{0,1}}{d\lambda} \cdot \frac{d\lambda}{dP} = \frac{dP_{0,1}}{d\lambda} \cdot \frac{1}{\frac{dP}{d\lambda}} = \boxed{\frac{f_{0,1}}{f_{cauchy}}}$$

Additional Proofs:

Show $P_{\mu,1} << \lambda$ $P_{\mu,1} = N(\mu, 1)$. ?

$P_{\mu,1}(A) = \int_A f_{\mu,1}\, d\lambda$ $f_{\mu,1}$ Exists and $f_{\mu,1} > 0$.

Now $\lambda(A) = 0 \Rightarrow P_{\mu,1}(A) = 0$ By Abs. cont. of the Integral (ACI). So $P_{\mu,1} << \lambda$ □

Show $P_{\mu,\sigma^2} << \lambda$ $P_{\mu,\sigma^2} = N(\mu, \sigma^2)$

$P_{\mu,\sigma^2} = \int_A f_{\mu,\sigma^2}\, d\lambda$; f_{μ,σ^2} Exists and $f_{\mu,\sigma^2} > 0$

$\lambda(A) = 0 \Rightarrow P_{\mu\sigma^2}(A) = 0$. By (ACI)

Show $P << \lambda$ P = Cauchy Dist.

$P(A) = \int_A f_{Cauchy}\, d\lambda$; f_{Cauchy} Exists and $f_{Cauchy} > 0$.

Now $\lambda(A) = 0 \Rightarrow P(A) = 0$ By (ACI.)

Show $P_{01} << \lambda$: $P_{01}(A) = \int_A f_{0,1}\, d\lambda$ ($f_{0,1}$ Exists and >0) Now $\lambda(A) = 0 \Rightarrow P_{01}(A) = 0$. (By ACI)

Show $\lambda << P_{01}$

$\boxed{\text{claim: } \lambda(A) = \int_A \frac{1}{f_{0,1}}\, dP_{0,1}}$ $f_{0,1}$ Exists & $f_{0,1} > 0$

Proof:
$\lambda(A) = \int_A \frac{1}{f_{0,1}} \frac{dP_{01}}{d\lambda}\, d\lambda$ as $P_{0,1} << \lambda$ & change of measures.

$= \int_A 1\, d\lambda = \lambda(A)$. □

$\lambda(A) = \int_A \frac{1}{f_{01}} \frac{dP_{01}}{d\lambda}\, d\lambda = \int_A \frac{1}{f_{0,1}} f_{0,1}\, d\lambda$

Now
$\lambda(A) = \int_A \frac{1}{f_{0,1}}\, dP_{0,1}$ since $f_{0,1} > 0$ $P_{01}(A) = 0 \Rightarrow \lambda(A) = 0$ By ACI

So $\lambda << P_{01}$ □.

Show $\lambda << P$ claim: $\lambda(A) = \int_A \frac{1}{f_{Cauchy}}\, dP$ (f_{Cauchy} Exists and is >0)

As in the above case, we can ~~state~~
claim is satisfied ~~com~~ similarly as above).

Now $P(A) = 0 \Rightarrow \lambda(A) = 0$ and $\lambda << P$
By (ACI)

Homework 2 - STA 5447

$\boxed{\text{Exercise \# 3}}$

(a) If $X \geq$ has df F then : $EX = \int_0^\infty (1-F(x))\,dx$.

$(\Omega, \mathcal{A}, P) \times (\mathbb{R}^+, \mathcal{B}_{\mathbb{R}^+}, \lambda) \leftarrow$ product measure space

$\boxed{d\lambda = d\lambda(t) = dt}$ since λ is the Lebesgue measure

$$EX = \int_\Omega X\,dP = \int_\Omega \int_0^X d\lambda\,dP =$$

$$= \int_\Omega \int_0^\infty 1_{[0<t<X]}\,dt\,dP$$

Note : $1_{[0<t<X]} = 1_{[0 \leq t \leq X]}$ since

$\int_{(0,X)} d\lambda = \int_{[0,X]} d\lambda$ $\left\{ \begin{array}{l} F(0)=0 \\ F \text{ is right continuous} \end{array} \right.$

$$\underset{\text{Tonelli}\ (1_{[0<t<X]} \geq 0)}{\overset{\text{by}}{=}} \int_0^\infty \int_\Omega 1_{[0<t<X]}\,dP\,d\lambda =$$

$$= \int_0^\infty P(X>t)\,dt = \int_0^\infty (1-F(t))\,dt \overset{t=x}{=} \int_0^\infty (1-F(x))\,dx$$

(b) If $X \geq 0$ has df F then: $EX^r = r \int_0^\infty x^{r-1}(1-F(x))\,dx$; $r > 1$.

$(\Omega, \mathcal{A}, P) \times (\mathbb{R}^+, \mathcal{B}_{\mathbb{R}^+}, \lambda)$ ← product measure space

$$EX^r = \int_\Omega X^r dP = \int_\Omega \int_0^X r t^{r-1} d\lambda\, dP =$$

$$= r \int_\Omega \int_{\mathbb{R}^+} t^{r-1} 1_{[0<t<X]} \, d\lambda\, dP \underset{\text{Note } t^{r-1} 1_{[0<t<X]} \geq 0}{=}$$

$$\underset{\text{Tonelli}}{\overset{\text{by}}{=}} r \int_{\mathbb{R}^+} t^{r-1} \left(\int_\Omega 1_{[0<t<X]} \, dP \right) d\lambda =$$

$$= r \int_{\mathbb{R}^+} t^{r-1} P(X>t)\, d\lambda \underset{t=x}{=}$$

$$= r \int_0^\infty x^{r-1} (1 - F(x))\, dx .$$

ⓒ If $E|X|<\infty$ then $EX = -\int_{-\infty}^{0} F(x)\,dx + \int_{0}^{\infty}(1-F(x))\,dx$

$(\Omega, \mathcal{A}, P) \times (\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \lambda)$ ← product measure space

$EX = EX^+ - EX^-$ well defined since $EX^+ \leq E|X| < \infty$, $EX^- \leq E|X| < \infty$ ‖ Note: $|X| = X^+ + X^-$. So $X^+ \leq |X|$ and $X^- \leq |X|$.

$$EX^- = \int_\Omega X^- dP = \int_\Omega \int_{-X^-}^{0} d\lambda\, dP = \int_\Omega \int_{\mathbb{R}^-} 1_{[-X^-<t<0]}\, d\lambda\, dP \underset{1_{[-X^-<t<0]}\geq 0}{\overset{\text{Tonelli}}{=}}$$

$$= \int_{\mathbb{R}^-} \int_\Omega 1_{[-X^-<t<0]}\, dP\, d\lambda = \int_{\mathbb{R}^-} P(-X^-<t)\, d\lambda \underset{(*)}{=} \int_{\mathbb{R}^-} P(X<t)\, d\lambda$$

(*) Claim that $\{\omega : -X^-(\omega)<t\} = \{\omega : X(\omega)<t\}$; $t<0$. Hence $P(-X^-<t) = P(X<t)$

Take an ω_0 such that $\boxed{X(\omega_0)<t}$. Then $X(\omega_0) = X^+(\omega_0) - X^-(\omega_0) < t$

Hence $-X^-(\omega_0) < t - X^+(\omega_0) \leq t+0 = t$

So $\boxed{-X^-(\omega_0) < t}$

Now take an ω_0 such that $\boxed{-X^-(\omega_0)<t}$. Recall $X^-(\omega_0) = \begin{cases} -X(\omega_0) & \text{if } X(\omega_0)\leq 0 \\ 0 & \text{ow} \end{cases}$, $-X^-(\omega_0) = \begin{cases} X(\omega_0) & \text{if } X(\omega_0)\leq 0 \\ 0 & \text{ow} \end{cases}$. $\Rightarrow$ So if $-X^-(\omega_0)<t$ then $\boxed{X(\omega_0)<t}$

Hence $EX^- = \int_{-\infty}^{0} P(X<t)\, d\lambda = \int_{-\infty}^{0} F(x)\, dx$.

$$EX^+ = \int_\Omega X^+ dP = \int_\Omega \int_0^{X^+} d\lambda\, dP = \int_\Omega \int_{\mathbb{R}^+} 1_{[0<t<X^+]}\, d\lambda\, dP \underset{1_{[0<t<X^+]}\geq 0}{\overset{\text{Tonelli}}{=}}$$

$$= \int_{\mathbb{R}^+} P(X^+>t)\, d\lambda \underset{(**)}{=} \int_{\mathbb{R}^+} P(X>t)\, d\lambda = \int_0^\infty (1-F(x))\, dx$$

(**) Claim that $\{\omega : X^+(\omega)>t\} = \{\omega : X(\omega)>t\}$; $t>0$. Hence $P(X^+>t) = P(X>t)$.

Take an ω_0 s.t. $\boxed{X(\omega_0)>t}$. Then $\boxed{X^+(\omega_0)>t}$ trivially because $X^+(\omega_0) = \begin{cases} X(\omega_0) & \text{if } X(\omega_0)\geq 0 \\ 0; & \text{ow} \end{cases}$.

Now take an ω_0 s.t. $\boxed{X^+(\omega_0)>t}$. Now $X(\omega_0) = X^+(\omega_0) - X^-(\omega_0) > t - 0 = t \Rightarrow \boxed{X(\omega_0)>t}$

$$EX = EX^+ - EX^- = \int_0^\infty (1-F(x))\,dx - \int_{-\infty}^{0} F(x)\,dx =$$

$$= -\int_{-\infty}^{0} F(x)\,dx + \int_0^\infty (1-F(x))\,dx.$$

HW 2

4.(a). Need to prove $EX = \sum_{k=1}^{\infty} P(X \geq k)$, $X \geq 0$

$$EX \overset{3(a)}{=} \int_0^{\infty} (1-F_X(x))dx = \sum_{k=0}^{\infty} \int_{[k,k+1)} (1-F_X(x))\,dx$$

$F_X(x)$ c.d.f. $F_X(x)=F(k)$ on $[k,k+1)$

$$= \sum_{k=0}^{\infty} (1-F(k))$$

$$= \sum_{k=0}^{\infty} P(X>k) = \sum_{k=0}^{\infty} P(X \geq k+1) = \sum_{k'=1}^{\infty} P(X \geq k')$$

x only take integer value. $k' = k+1$

(b). $EX^2 = \sum_{k=0}^{\infty} (2k+1) P(X>k)$

We use $EX^r = \int_0^{\infty} r t^{r-1} (1-F(t))\,dt$

let $r=2$ $\Rightarrow$ $EX^2 = \int_0^{\infty} 2t(1-F(t))\,dt$

$$= \sum_{k=0}^{\infty} \int_{[k,k+1)} 2t(1-F(t))dt$$

$$= \sum_{k=0}^{\infty} (1-F(k)) \int_{[k,k+1)} 2t\,dt$$

$$= \sum_{k=0}^{\infty} (1-F(k))(2k+1)$$

$$= \sum_{k=0}^{\infty} P(X>k)(2k+1)$$

All reasoning is similar to (a)

5. ① We first show $\int_0^1 \left(\int_1^\infty f(x,y)\,dy\right)dx = \int_0^1 \frac{1}{x}(e^{-x}-e^{-2x})\,dx$

exists and >0

where $f(x,y) = e^{-xy} - 2e^{-2xy}$.

First it's easy to see $\frac{1}{x}(e^{-x}-e^{-2x}) > 0$

since $e^x > 1$ for $x \in (0,1]$.

Now we prove the integral exists $(<\infty)$

Let $g(x) = \begin{cases} \frac{1}{x}(e^{-x}-e^{-2x}) \\ 1 \end{cases}$ by L'Hospital's Rule as $x \to 0$

we know that $g(x) \in C^1([0,1])$

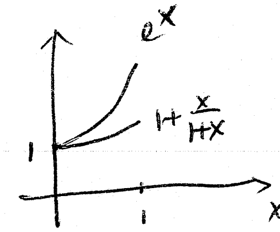
we will show $g(x) \leq 1$ on $[0,1]$

Since $g'(x) = \frac{e^{-2x}}{x^2}\left((2x+1) - (x+1)e^x\right)$

and $e^x \geq \frac{2x+1}{x+1} = 1 + \frac{x}{x+1}$ on $[0,1] \Rightarrow g'(x) \leq 0$

$\uparrow$

	$x=0$	$x=1$
e^x	1	e
$1+\frac{x}{1+x}$	1	$3/2$



e^x & $1+\frac{x}{1+x}$ both strictly $\nearrow$

Therefore $g(x) \downarrow$ on $[0,1]$ with $g(0)$ max

$\Rightarrow g(x) \leq 1 \Rightarrow \int_0^1 g(x)\,dx = \int_0^1 \frac{1}{x}(e^{-x}-e^{-2x})\,dx \leq 1 < \infty$

② Now we show

$$\int_1^\infty \left(\int_0^1 f(x,y)\,dx \right) dy = \int_1^\infty \frac{1}{y} \left(e^{-2y} - e^{-y} \right) dy$$

exists and < 0

As before, it's easy to see $\frac{1}{y} e^{-2y} - e^{-y} < 0$ on $(1, \infty)$

We want $\int_1^\infty \left| \frac{1}{y}(e^{-2y} - e^{-y}) \right| dy < \infty$

Since

$$\int_1^\infty \frac{1}{y} e^{-2y}\,dy \le \int_1^\infty e^{-2y}\,dy < \infty$$

$$\int_1^\infty \frac{1}{y} e^{-y}\,dy \le \int_1^\infty e^{-y}\,dy < \infty$$

exponential kernel

$$\Rightarrow \int_1^\infty \left| \frac{1}{y}(e^{-2y} - e^{-y}) \right| dy$$

$$\le \underbrace{\int_1^\infty \frac{1}{y} e^{-2y}\,dy}_{>0} + \underbrace{\int_1^\infty \frac{1}{y} e^{-y}\,dy}_{>0} < \infty$$

③ Now we show that above results don't contradict Tonelli's Theorem.

We will prove that

a). $f(x,y) \not\geq 0$, b). $\int \left(\int |f(x,y)|\,dx \right) dy = \infty$.

c). $\int \left(\int |f(x,y)|\,dy \right) dx = \infty$.

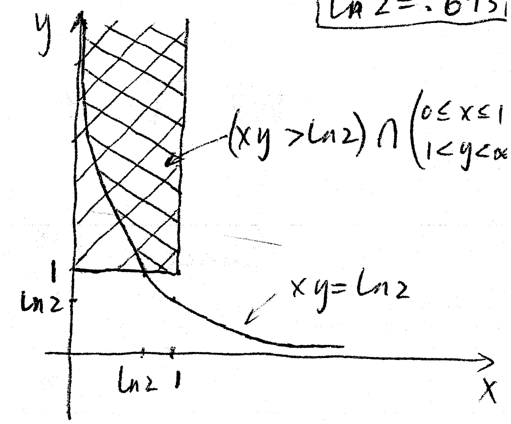
Now,

a). $f(x,y) = e^{-xy} - 2e^{-2xy}$, $x \in [0,1]$, $y \in (1,\infty)$

Let $x \to 0^+$, $y \to 1^+$ $\Rightarrow$ $f(x,y) \to -1 < 0$.

b). prove $\int\left(\int |f(x,y)|\,dx\right)dy = \infty$

$\boxed{\ln 2 = .6931}$



$$\int\left(\int |f(x,y)|\,dx\right)dy$$

$$= \int_1^\infty \left(\int_0^1 \left| e^{-xy} - 2e^{-2xy} \right| dx \right) dy$$

$$= \int_1^\infty \left(\int_{\frac{\ln 2}{y}}^1 \left(e^{-xy} - 2e^{-2xy}\right) dx \right) dy$$

$$+ \int_1^\infty \left(\int_0^{\frac{\ln 2}{y}} \left(-e^{-xy} + 2e^{-2xy}\right) dx \right) dy$$

$$= \int_1^\infty \left[\frac{1}{y}\left(e^{-2y} - e^{-y}\right) + \frac{1}{4y} \right] dy + \int_1^\infty \frac{1}{4y}\,dy$$

$$= \underbrace{\int_1^\infty \left[\frac{1}{y}\left(e^{-2y} - e^{-y}\right) \right] dy}_{<\infty} + \underbrace{\int_1^\infty \frac{1}{2y}\,dy}_{\hookrightarrow\ \frac{1}{2}\ln y \big|_1^\infty \to \infty}.$$

c). Prove $\int\left(\int |f(x,y)|\,dy\right)dx = \infty$.

Zhi

$$\int\left(\int|f(x,y)|\,dy\right)dx$$

$$=\int_0^1\left(\int_1^\infty|f(x,y)|\,dy\right)dx$$

$$=\int_0^{\ln 2}\left(\int_1^{\frac{\ln 2}{x}}(-f(x,y))\,dy\right)dx+\int_0^{\ln 2}\left(\int_{\frac{\ln 2}{x}}^\infty f(x,y)\,dy\right)dx$$

$$+\int_{\ln 2}^1\left(\int_1^\infty f(x,y)\,dy\right)dx$$

$$=\int_0^{\ln 2}\left(\frac{1}{x}e^{-2x}-\frac{1}{x}e^{-x}+\frac{1}{4x}\right)dx+\int_0^{\ln 2}\frac{1}{4x}\,dx$$

$$+\int_{\ln 2}^1\left(\frac{1}{x}e^{-x}-\frac{1}{x}e^{-2x}\right)dx$$

$$=\underbrace{-\int_0^{\ln 2}\frac{1}{x}(e^{-x}-e^{-2x})\,dx}_{a}+\underbrace{\int_{\ln 2}^1\frac{1}{x}(e^{-x}-e^{-2x})\,dx}_{b}$$

$$+\underbrace{\int_0^{\ln 2}\frac{1}{2x}\,dx}_{c}$$

$$|a|\le\int_0^1\frac{1}{x}(e^{-x}-e^{-2x})\,dx<\infty$$

$$|b|\le\int_0^1\frac{1}{x}(e^{-x}-e^{-2x})\,dx<\infty.$$

since $\frac{1}{x}(e^{-x}-e^{-2x})>0$

$$|c|=\left|\frac{1}{2}\ln x\Big|_0^{\ln 2}\right|=\infty.$$

Q.E.D

HOMEWORK 2

4.2.1] μ and ν are σ-finite measures on $(\Omega, \mathcal{A})$

Φ and Ψ are σ-finite signed meas on $(\Omega, \mathcal{A})$

a) If $\Phi << \mu$ and $\Psi << \mu$ then $\dfrac{d(\Phi+\Psi)}{d\mu} = \dfrac{d\phi}{d\mu} + \dfrac{d\Psi}{d\mu}$ a.e μ.

Sol^n (i) $\Phi << \mu$ and $\Psi << \mu$ so $\dfrac{d\Phi}{d\mu}$ and $\dfrac{d\Psi}{d\mu}$ are the RN deriv.

If $\mu(A) = 0 \rightarrow \Phi(A) = 0$ and $\Psi(A) = 0$

$\rightarrow \Phi(A) + \Psi(A) = 0 \rightarrow (\Phi+\Psi)(A) = 0$

$\therefore \Phi + \Psi << \mu \rightarrow \dfrac{d(\Phi+\Psi)}{d\mu}$ is the RN derivative

(ii) By the RN Theorem: $\displaystyle\int_A \left[\frac{d(\Phi+\Psi)}{d\mu}\right] d\mu = (\Phi+\Psi)(A)$

$$= \Phi(A) + \Psi(A)$$

$$= \int_A \frac{d\Phi}{d\mu}\, d\mu + \int_A \frac{d\Psi}{d\mu}\, d\mu \quad \left\{ \begin{array}{l} \text{RN} \\ \text{theorem} \end{array} \right.$$

$$\therefore \int_A \frac{d(\phi+\psi)}{d\mu} = \int_A \frac{d\phi}{d\mu} + \int_A \frac{d\psi}{d\mu} = \int_A \frac{d\phi}{d\mu} + \frac{d\psi}{d\mu} \quad ; \forall A \in \mathcal{A}$$

$$\Rightarrow \frac{d(\phi+\psi)}{d\mu} = \frac{d\phi}{d\mu} + \frac{d\psi}{d\mu} \quad \text{a.e } \mu$$

(b) If $\phi << \mu$ and $\mu << \nu$ then $\frac{d\phi}{d\nu} = \frac{d\phi}{d\mu}\frac{d\mu}{d\nu}$ a.e ν

(i) Since $\phi << \mu$ and $\mu << \nu$; $\frac{d\phi}{d\mu}$ and $\frac{d\mu}{d\nu}$ are the RN derivat is

Now $\nu(A) = 0 \Rightarrow \mu(A) = 0 \Rightarrow \Phi(A) = 0$ for any $A \in \mathcal{A}$

$\therefore$ $\phi << \nu$ and $\frac{d\phi}{d\nu}$ is the RN derivative of ϕ wrt ν

(ii)

Since $\Phi << \mu$ by the RN theorem we have

$$\Phi(A) = \int_A \frac{d\phi}{d\mu} \cdot d\mu$$

$$= \int_A \frac{d\phi}{d\mu} \frac{d\mu}{d\nu} d\nu$$

$\left\{ \because \mu << \nu \text{ and } \int_A \frac{d\phi}{d\mu} \cdot d\mu \right.$ is well defined we apply the change of var theorem

Since $\Phi << \nu \Rightarrow \Phi(A) = \int_A \frac{d\phi}{d\nu} d\nu$

$$\therefore \int_A \frac{d\phi}{d\nu} d\nu = \int_A \frac{d\phi}{d\mu}\frac{d\mu}{d\nu} d\nu \quad \text{for any } A \in \mathcal{A}$$

$$\Rightarrow \frac{d\phi}{d\nu} = \frac{d\phi}{d\mu}\frac{d\mu}{d\nu} \quad \text{a.e } \nu$$

{ by RN theorem - uniqueness

4.2.2] P_{μ,σ^2} denotes $N(\mu,\sigma^2)$ distribution. P is Cauchy.

a) Show $P_{\mu,1} << P_{0,1}$ and compute $dP_{\mu,1}/dP_{0,1}$

$$P_{\mu,\sigma^2}(A) = \int_A \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu^2)\right\} d\lambda(x) = \int_A f_{\mu,\sigma^2}(x)\,dx$$

$$P(A) = \int_A \frac{1}{(2\pi)} \frac{1}{(1+x^2)} d\lambda(x)$$

Now $$P_{\mu,1}(A) = \int_A f_{\mu,1}(x)\,dx = \int_A \frac{dP_{\mu,1}}{d\lambda}\,d\lambda$$ {and must show $P_{\mu,1} << \lambda$

$$= \int_A \frac{f_{\mu,1}(x)}{f_{0,1}(x)} f_{0,1}(x)\,dx$$

{ must Justify change of variable theorem since $\lambda << P_{0,1}$

$$= \int_A \frac{f_{\mu,1}(x)}{f_{0,1}(x)}\,dP_{0,1}(x)$$

If $P_{0,1}(A) = 0 \Rightarrow dP_{0,1}(A) = 0 \rightarrow P_{\mu,1}(A) = 0$

so $P_{\mu,1} << P_{0,1}$

Now $$\frac{dP_{\mu,1}}{dP_{0,1}} = \frac{dP_{\mu,1}}{d\lambda}\frac{d\lambda}{dP_{0,1}}$$ a.e λ from Ex 4.2.1

$$= \left(\frac{dP_{\mu,1}}{d\lambda}\right)\frac{1}{\left(\frac{dP_{0,1}}{d\lambda}\right)}$$ a.e λ

$$= \frac{f_{\mu,1}}{f_{0,1}} = \frac{\exp\{-\frac{1}{2}(x-\mu)^2\}}{\exp\{-\frac{1}{2}(x)^2\}}$$

b) $$P_{0,\sigma^2}(A) = \int_A f_{0,\sigma^2}(x)\,dx$$

$$= \int_A \frac{f_{0,\sigma^2}(x)}{f_{0,1}(x)} f_{0,1}(x)\,dx$$

$$= \int_A \frac{f_{0,\sigma^2}(x)}{f_{0,1}(x)}\,dP_{0,1}$$

If $P_{0,1}(A) = dP_{0,1}(A) = P_{0,\sigma^2}(A) = 0$

$\therefore\ P_{0,\sigma^2} << P_{0,1}$

Now $$\frac{dP_{0,\sigma^2}}{dP_{0,1}} = \frac{dP_{0,\sigma^2}}{d\lambda}\,\frac{d\lambda}{dP_{0,1}} = \left(\frac{dP_{0,\sigma^2}}{d\lambda}\right)\frac{1}{\left(\frac{dP_{0,1}}{d\lambda}\right)} \quad \text{a.e } \lambda$$

$$= \frac{f_{0,\sigma^2}}{f_{0,1}} = \frac{1/\sigma^2\ \exp\left\{-\frac{1}{2\sigma^2}x^2\right\}}{\exp\left\{-\frac{1}{2}x^2\right\}}$$

(c) To show $P << P_{0,1}$

$$P(A) = \int_A \frac{1}{2\pi}\,\frac{1}{(1+x^2)}\,dx = \int_A g(x)\,dx$$

$$= \int_A \frac{g(x)}{f_{0,1}(x)} f_{0,1}(x)\,dx = \int_A \frac{g(x)}{f_{0,1}(x)}\,dP_{0,1}$$

So if $P_{0,1}(A) = 0 \Rightarrow P(A) = 0$

Alternately: $f_{0,1} \geq 0$ always

$P_{0,1}(A)=0 \Rightarrow \int_A f_{0,1}(x)\,d\lambda(x) = 0 \Rightarrow \lambda(A) = 0 \Longrightarrow P(A) = 0 \quad \{\because P << \lambda\}$

$\therefore\ P << P_{0,1}$ so $\dfrac{dP}{dP_{0,1}} = \dfrac{dP}{d\lambda}\dfrac{d\lambda}{dP_{0,1}}$ a.e λ

$$= \left(\frac{dP}{d\lambda}\right)\frac{1}{\left(\frac{dP_{0,1}}{d\lambda}\right)} = \frac{g(x)}{f_{0,1}(x)}$$

Now we will show $P_{0,1} << P$

$g(\cdot) \geq 0$ always so $P(A)=0 \Rightarrow \int_A g(x)\,d\lambda(x) = 0 \Rightarrow \lambda(A)=0$

$\Rightarrow P_{0,1}(A)=0$ $\{\because P_{0,1} << \lambda\}$

$\therefore\ P_{0,1} << P$

Now $\dfrac{dP_{0,1}}{dP} = \dfrac{dP_{0,1}}{d\lambda}\dfrac{d\lambda}{dP} = \left(\dfrac{dP_{0,1}}{d\lambda}\right)\dfrac{1}{\left(\frac{dP}{d\lambda}\right)} = \dfrac{f_{0,1}(x)}{g(x)}$

To show if $\mu << \lambda$ & $\lambda << \mu$ then $\dfrac{d\mu}{d\lambda}\dfrac{d\lambda}{d\mu} = 1$ a.e

Now $\displaystyle\int_A \frac{d\mu}{d\lambda}\frac{d\lambda}{d\mu}\,d\mu = \int_A \frac{d\mu}{d\lambda}\,d\lambda$ $\{$ Change of var. theorem $\because \lambda << \mu$

$= \mu(A)$ $\{\because$ RN Theorem

$\displaystyle\int \frac{d\mu}{d\lambda}\frac{d\lambda}{d\mu}\,d\lambda = \int \frac{d\lambda}{d\mu}\,d\mu$ $\{$ $\mu << \lambda$ & change of var. theor

$= \lambda(A)$ $\{\because$ R-N Theorem

$\therefore\ \displaystyle\int_A \frac{d\mu}{d\lambda}\frac{d\lambda}{d\mu}\,d\mu = \mu(A) = \int_A 1\,d\mu \Rightarrow \frac{d\mu}{d\lambda}\frac{d\lambda}{d\mu} = 1$ a.e λ

$\displaystyle\int_A \frac{d\mu}{d\lambda}\frac{d\lambda}{d\mu}\,d\lambda = \lambda(A) = \int_A 1\,d\lambda \Rightarrow \frac{d\mu}{d\lambda}\frac{d\lambda}{d\mu} = 1$ a.e μ

3] If $X \geq 0$ with df F then show :

(a) $E(X) = \int_0^\infty (1-F(x))\, dx$

(b) $E(X^r) = \int_0^\infty r x^{r-1} (1-F(x))\, dx$ for $r > 1$.

(c) If $E|X| < \infty$ the $E(X) = -\int_{-\infty}^0 F(x)\, dx + \int_0^\infty (1-F(x))\, dx$

(a) $X : (\Omega, \mathcal{A}, P) \longrightarrow (R, \mathcal{B})$

$$X = \int_0^X 1\, d\lambda$$

$$E(X) = \int_\Omega X(\omega)\, dP(\omega)$$

$$= \int_\Omega \int_0^{X(\omega)} d\lambda(t)\, dP(\omega)$$

$$= \int_\Omega \int_0^\infty I_{[0 < t < X(\omega)]}\, d\lambda(t)\, dP(\omega)$$

This is defined on the product space $(\Omega \times R ; \mathcal{A} \times \mathcal{B})$ with product measure. $\phi \equiv P \times \lambda$

We now apply Tonelli's theorem for which we need to check

$\int\int |X| d\mu\, d\nu < \infty$ OR $\int\int |X|\, d\nu\, d\mu < \infty$ or $X \geq 0$

For our problem $X \equiv I_{[0<t<X]}$ which is always ≥ 0

$$\therefore \quad E(X) = \int_0^{\infty} \int_{\Omega} I_{[0<t<X]} \, dP \, d\lambda(t)$$

$$= \int_0^{\infty} P(X>t) \, dt$$

$$= \int_0^{\infty} (1-F(t)) \, dt$$

(b) $$X^r = \int_0^X r t^{r-1} dt \qquad ; \; r>1$$

$$E(X^r) = \int_{\Omega} X^r dP$$

$$= \int_{\Omega} \int_0^X r t^{r-1} dt \, dP$$ on the space $(\Omega \times \mathbb{R}, \mathcal{A} \times \mathcal{B})$ with meas $\Phi = P \times \lambda$

$$= \int_{\Omega} \int_0^{\infty} r t^{r-1} I_{[0<t<X]} \, dt \, dP$$

Now $r t^{r-1} I_{[0<t<X]} \geq 0$ $\quad \{$ since $t>0$, $r>1$

$\therefore$ by Tonelli's theorem

$$E(X^r) = \int_0^{\infty} \int_{\Omega} r t^{r-1} I_{[0<t<X]} \, dP \, dt$$

$$= \int_0^\infty r t^{r-1} \int_\Omega I_{[X>t]} \, dP \quad dt$$

$$= \int_0^\infty r t^{r-1} P(X>t) \, dt$$

$$= \int_0^\infty r t^{r-1} [1 - F(t)] \, dt$$

(c) $X = X^+ - X^-$

$\therefore$ $E(X) = E(X^+) - E(X^-)$ $\left\{ \begin{array}{l} \text{Here since } E|X| < \infty \Rightarrow E(X^+) < \infty \\ \text{and } E(X^-) < \infty \text{ so we will never} \\ \text{have situations like } \infty - \infty \end{array} \right.$

Now $X^+ \geq 0$ always $\Rightarrow$ $E(X^+) = \int_0^\infty P(X^+ > t) \, dt$

$X^- \geq 0$ always $\Rightarrow$ $E(X^-) = \int_0^\infty P(X^- > t) \, dt$

Now $\left. \begin{array}{l} X^+ = X \text{ when } X \geq 0 \\ 0 \text{ ow} \end{array} \right\}$ $P(X^+ > t) = P(X > t) + P(0 > t)$

$= P(X>t) + 0$ $\{\because t \geq 0\}$

$\left. \begin{array}{l} X^- = -X \text{ when } X \leq 0 \\ 0 \text{ ow} \end{array} \right\}$ $P(X^- > t) = P(-X > t) + P(0 > t)$

$= P(-X > t)$

$$\therefore E(X) = \int_0^\infty P(X>t) \, dt + \int_0^\infty P(-X > t) \, dt$$

$$= \int_0^\infty P(X>t) \, dt + \int_{-\infty}^0 -P(X \leq x) \, dx$$ {change of variable

$$E(X) = \int_0^{\infty} P(X > x)\,dx - \int_{-\infty}^{0} P(X \leq x)\,dx$$

$$= \int_0^{\infty} [1 - F(x)]\,dx - \int_{-\infty}^{0} F(x)\,dx$$

4] $X \geq 0$, integer valued discrete

$$E(X) = \int_0^{\infty} [1 - F(x)]\,dx \qquad \{ \text{ from } 3(a)$$

$$= \sum_{k=0}^{\infty} \int_{[k,k+1)} [1 - F(x)]\,dx$$

Since X is discrete integer valued on $[k, k+1)$ the value of $F(x)$ is a constant $= F(k)$

$$\therefore \quad E(X) = \sum_{k=0}^{\infty} \int_{[k,k+1)} [1 - F(k)]\,dx = \sum_{k=0}^{\infty} [1 - F(k)] \int_{[k,k+1)} dx$$

$$= \sum_{k=0}^{\infty} [1 - F(k)]\,(k + 1 - k)$$

$$= \sum_{k=0}^{\infty} [1 - F(k)]$$

$$= \sum_{k=0}^{\infty} P(X > k)$$

$$= \sum_{k=0}^{\infty} P(X \geq k + 1)$$

$$= \sum_{t=1}^{\infty} P(X \geq t) \qquad \{ \text{ Put } t \equiv k + 1$$

Now $E(X^2) = \int_0^\infty 2t\,(1-F(t))\,dt$ $\Big\{$ from 3(b) with $r=2$.

$$= \sum_{k=0}^{\infty} \int_{[k,k+1)} 2t\,(1-F(t))\,dt$$

Again ! X is discrete so $F(t)$ is constant on $[k,k+1)$ and takes the value $F(k)$

$$\therefore\ E(X^2) = \sum_{k=0}^{\infty} \int_{[k,k+1)} 2t\,[1-F(k)]\,dt$$

$$= \sum_{k=0}^{\infty} [1-F(k)] \int_{[k,k+1)} 2t\,dt$$

$$= \sum_{k=0}^{\infty} [1-F(k)] \left(t^2 \Big|_k^{k+1} \right)$$

$$= \sum_{k=0}^{\infty} [1-F(k)] \left((k+1)^2 - k^2 \right)$$

$$= \sum_{k=0}^{\infty} [1-F(k)] \,(2k+1)$$

$$= \sum_{k=0}^{\infty} P(X>k)\,(2k+1)$$

5] $A_1 = [0,1]$ $A_2 = (1,\infty)$

$f(x,y) = e^{-xy} - 2e^{-2xy}$.

a) $\int_0^1 \int_1^\infty f(x,y)\,dy\,dx = \int_0^1 \int_0^\infty \left(e^{-xy} - 2e^{-2xy}\,dy\right) dx$

$= \int_0^1 \frac{1}{x}(e^{-x} - e^{-2x})\,dx$

Now $\lim_{x\to 0} \frac{e^{-x} - e^{-2x}}{x} = \lim_{x\to 0} -e^{-x} + 2e^{-x} = 1 < \infty$

$\lim_{x\to 1} \frac{e^{-x} - e^{-2x}}{x} = e^{-1} - e^{-2} < \infty$

$\therefore$ $\frac{1}{x}(e^{-x} - e^{-2x})$ is continuous on $[0,1]$ and bdd on $[0,1]$

and hence it is integrable

Also $\left(\frac{e^{-x} - e^{-2x}}{x}\right) > 0$ a.e $\Rightarrow$ $\int_0^1 \frac{e^{-x} - e^{-2x}}{x} > 0$

b) $\int_1^\infty \int_0^1 \left(f(x,y)\,dx\right) dy = \int_1^\infty \frac{1}{y}(e^{-2y} - e^{-y})\,dy$

Now $\frac{e^{-2y} - e^{-y}}{y} < e^{-2y} - e^{-y}$ for $y \in (1,\infty)$

$\therefore$ $\int_1^\infty \left(\frac{e^{-2y} - e^{-y}}{y}\right) dy < \int (e^{-2y} - e^{-y})\,dy$

Now $\displaystyle\int_1^\infty e^{-2y}\, e^{-y}\, dy = \left(\frac{e^{-2y}}{-2}\right) - \left(\frac{e^{-y}}{-1}\right)\Bigg|_1^\infty$

$$= e^{-y} - \frac{e^{-2y}}{2}\Bigg|_1^\infty$$

$$= -e^{-1} + \frac{e^{-2}}{2}$$

$$= -\tfrac{1}{2} \qquad < \infty$$

$\therefore \quad \displaystyle\int_1^\infty \left(\frac{e^{-2y} - e^{-y}}{y}\right) dy \quad < \quad \int_1^\infty (e^{-2y} - e^{-y})\, dy \quad < \quad \infty$

Also $\quad \dfrac{e^{-2y} - e^{-y}}{y} < 0 \quad$ a.e $\quad$ on $(1, \infty)$.

$\Rightarrow \quad \displaystyle\int_1^\infty \left(\frac{e^{-2y} - e^{-y}}{y}\right) < 0$

So $\displaystyle\int_1^\infty \frac{1}{y}(e^{-2y} - e^{-y})\, dy$ exists and is < 0

No. it does not contradict Fubini or Tonelli's theorem. since the conditions of Tonelli's theorem are not satisfied

$f(x,y)$ is not always ≥ 0

$\displaystyle\iint |f(x,y)|\, dx\, dy \not< \infty$ and $\displaystyle\iint |f(x,y)|\, dy\, dx \not< \infty$

Now (i) $f(x,y) = e^{-xy} - 2e^{-2xy}$ $\quad x \in [0,1]$ and $y \in (1,\infty)$

Let $x \to 0^+$ & $y \to 1^+$ $\Rightarrow f(x,y) \to -1 < 0$

(ii) To show $\int\int |f(xy)|\, dx\, dy = \infty$

$$\int\int |f(x,y)|\, dx\, dy = \int_1^\infty \int_0^1 |e^{-xy} - 2e^{-2xy}|\, dx\, dy$$

$$= \int_1^\infty \int_0^{\ln 2/y} -(e^{-xy} - 2e^{-2xy})\, dx\, dy + \int_1^\infty \int_{\frac{\ln 2}{y}}^1 +(e^{-xy} - 2e^{-2xy})\, dx\, dy$$

$$= \int_1^\infty \left(\frac{e^{-xy}}{y} - \frac{\cancel{2}e^{-2xy}}{\cancel{2}y}\right)_0^{\log 2/y} dy + \int_1^\infty \left(\frac{e^{-xy}}{-y} - \frac{2e^{-2xy}}{-2y}\right)_{\frac{\log 2}{y}}^1 dy$$

$$= \int_1^\infty \left(\frac{1}{2y} - \frac{1}{4y}\right) dy + \int_1^\infty \left[\left(-\frac{e^{-y}}{y} + \frac{e^{-2y}}{y}\right) - \left(-\frac{1}{2y} + \frac{1}{4y}\right)\right] dy$$

$$= \int_1^\infty \left(\frac{1}{4y}\right) dy + \int_1^\infty \frac{1}{y}\left(e^{-2y} - e^{-y}\right) dy + \int_1^\infty \frac{1}{4y}\, dy$$

$$= \infty$$

(iii) $|||^{ly}$ we can show

$$\int_0^1 \int_1^\infty |e^{-xy} - 2e^{-2xy}|\, dy\, dx = \int_0^1 \int_1^{\frac{\log 2}{x}} -\left[e^{-xy} - 2e^{-2xy}\right] dy\, dx + \int_0^1 \int_{\frac{\log 2}{x}}^\infty +\left[e^{-xy} - 2e^{-2xy}\right] dy\, dx$$

$$= \int_0^{\log 2} \left(\frac{e^{-xy}}{x} - \frac{2e^{-2xy}}{2x}\right)_1^{\log 2/x} dx + \int_0^{\log 2} \left(\frac{e^{-xy}}{-x} - \frac{2e^{-xy}}{2x}\right)_{\log 2/x}^\infty dx + \int_{\log 2}^1 \int_1^\infty \left(\frac{1}{x}e^{-x} - \frac{1}{x}e^{-2x}\right)$$

Pg 66 Theorem 2.2 (use)

$\mu_1(A) = \int_A X \cdot d\mu_2$ then X is the density of μ_1 wrt μ_2

If we can write this (ie density exists) then $\mu_1 << \mu_2$

Ex 4.2.1 (Pg 67) μ and ν are 2 σ-finite measures on $(\Omega, \mathcal{A})$

ϕ and ψ are σ-finite meas on $(\Omega, \mathcal{A})$ Then

a) $\dfrac{d(\phi+\psi)}{d\mu} = \dfrac{d\phi}{d\mu} + \dfrac{d\psi}{d\mu}$ a.e μ if $\phi << \mu$ & $\psi << \mu$

b) $\dfrac{d\phi}{d\nu} = \dfrac{d\phi}{d\mu} \dfrac{d\mu}{d\nu}$ a.e ν if $\phi << \mu$ and $\mu << \nu$

Note : If $\mu_1 << \mu_2$ and $\mu_2 << \mu_1$ then $\dfrac{d\mu_2}{d\mu_1} \dfrac{d\mu_1}{d\mu_2} = 1$ a.e.

RN derivatives $\equiv$ functions.

a) Step 1 Verify if $\phi + \psi << \mu$ (then the RN derivative w.r.t μ makes sense)

Step 2 Use fact if $\int_A X(\omega) d\mu(\omega) = \int_A Y(\omega) d\mu(\omega)$ $\forall A \in \mathcal{A}$

then $X = Y$ a.e μ.

Check if $\dfrac{d(\phi+\nu)}{d\mu} = \dfrac{d\phi}{d\mu} + \dfrac{d\nu}{d\mu}$ a.e μ.

ie check $Z = X + Y$

Check $\int_A \dfrac{d(\phi+\nu)}{d\mu} \stackrel{?}{=} \int_A \dfrac{d\phi}{d\mu} + \int_A \dfrac{d\nu}{d\mu}$ $\forall A \in \mathcal{A}$.

2.1 (b) $\frac{d\phi}{d\nu} = \frac{d\phi}{d\mu}\frac{d\mu}{d\nu}$ a.e ν when $\phi << \mu$ & $\mu << \nu$.

Step 1 Show $\phi << \nu$ (then the RN derivative $\frac{d\phi}{d\nu}$ exists)

Step 2 Show $\int_A \frac{d\phi}{d\nu} d\nu = \int_A \frac{d\phi}{d\mu} \cdot \frac{d\mu}{d\nu}\, d\nu \quad \forall\ A \in \mathcal{A}$

$$\int_A \frac{d\phi}{d\mu}\frac{d\mu}{d\nu}\, d\nu = \int_A \frac{d\phi}{d\mu}\, d\mu = \phi(A)$$

because $\frac{d\phi}{d\mu}$ is RN derv. wrt μ.

use theorem 2.2 to show (and must check cond^n are satisfied)

Also $\int_A \frac{d\phi}{d\nu}\, d\nu = \phi(A)$ {since $\frac{d\phi}{d\nu}$ is RN derv. w.rt ν}

Corollary of Theorem 2.2

If $\mu << \lambda$ and $\lambda << \mu$

then $\frac{d\mu}{d\lambda}\frac{d\lambda}{d\mu} = 1$ a.e [wrt μ & λ]

$\Downarrow$

$\frac{d\mu}{d\lambda} = \frac{1}{\left(\frac{d\lambda}{d\mu}\right)}$ a.e [What happens at pt where $(d\lambda/d\mu) = 0$?]

Proof We need to show $\int_A \frac{d\mu}{d\lambda}\frac{d\lambda}{d\mu}\, d\mu = \int_A 1\, d\mu$

and $\int \frac{d\mu}{d\lambda}\frac{d\lambda}{d\mu}\, d\lambda = \int_A 1\, d\lambda$

$\forall\ A \in \mathcal{A}$

(i) Show $\int_A \frac{d\mu}{d\lambda}\frac{d\lambda}{d\mu}\cdot d\mu \underset{\uparrow}{=} \int_A^{X} \frac{d\mu}{d\lambda}\, d\lambda = \mu(A)$ $\hookrightarrow$ since $\frac{d\mu}{d\lambda}$ is RN derivative of μ wrt λ

use Theorem 2.2

(ii) Show

$$\int_A \frac{d\mu}{d\lambda}\frac{d\lambda}{d\mu}\, d\lambda \underset{\uparrow}{=} \int_A \frac{d\lambda}{d\mu}\, d\mu = \lambda(A)$$

use theorem 2.2.

Ex 2.2 $\quad P_{\mu\,\sigma^2}$ denotes $N(\mu,\sigma^2)$ dist$^{\underline{n}}$. [ie induced meas of a $N(\mu,\sigma^2)$ r.v

P is cauchy.

ie $P_{\mu,\sigma^2}(A) = \int_A \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right\} \cdot d\lambda \equiv dx$

$$P(A) = \int_A \frac{1}{2\pi}\frac{1}{1+x^2}\, dx$$

We know $P_{\mu,\sigma^2} << \lambda$

and $P << \lambda$

(a) To show $P_{\mu,1} << P_{0,1}$ ie if $P_{0,1}(A)=0 \Rightarrow P_{\mu,1}(A)=0$

Now $P_{\mu,1}(A) = \int_A \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}(x-\mu)^2\right\} \cdot dx$

$$= \int_A f_{\mu,1}(x)\, dx$$

$$= \int_A \left[\frac{f_{\mu,1}(x)}{f_{0,1}(x)}\right] f_{0,1}(x) \quad dx$$

$$= \int_A \frac{f_{\mu,1}(x)}{f_{0,1}(x)} \, dP_{0,1}$$

so $P_{0,1}(A) = 0 \Rightarrow P_{\mu,1}(A) = 0$

[ALT : $P_{0,1}(A) = 0 \xRightarrow{\text{show}} \lambda(A) = 0 \xRightarrow{\text{use a.continuity}} P_{\mu,1}(A) = 0.$

$\downarrow$

$\int_A X \, d\lambda = 0$ & $X > 0 \Rightarrow \lambda(A) = 0$]

To compute $\quad \dfrac{dP_{\mu,1}}{dP_{0,1}} = \dfrac{d \cdot P_{\mu,1}}{d\lambda} \dfrac{d\lambda}{dP_{0,1}}$ (from 4.2.1)

$$= \left(\frac{dP_{\mu,1}}{d\lambda}\right) \frac{1}{\left(\frac{dP_{0,1}}{d\lambda}\right)} \quad \text{a.e. } \lambda$$

$$= \frac{(dP_{\mu,1}/d\lambda)}{(dP_{0,1}/d\lambda)} \quad \text{a.e } \lambda$$

$$= \frac{f_{\mu,1}}{f_{0,1}}$$

4.2.2 (b) Show $P_{0,\sigma^2} << P_{0,1}$ and compute $dP_{0,\sigma^2} / dP_{0,1}$ (3)

Same arguement as part (a)

(c) compute $\frac{dP}{dP_{0,1}}$ & $\frac{dP_{0,1}}{dP}$

<u>Show</u> $P << P_{0,1}$

① use $P << \lambda << P_{0,1}$ [must argue clearly)

② Alt use $f = \frac{\left(\frac{1}{\pi} \frac{1}{1+x^2}\right)}{\left(\frac{1}{\sqrt{2\pi}} e^{-x^2/2\sigma^2}\right)}$

show $P(A) = \int_A f \cdot dP_{0,1}$

For $\frac{dP_{0,1}}{dP} = \frac{1}{\left(\frac{dP}{dP_{0,1}}\right)}$ but we must prove $P_{0,1} << P$

#3]

If $X \geq 0$ with df F

$$(\Omega, \mathcal{A}, P) \xrightarrow{X} (\mathbb{R}, \mathcal{B}, \lambda)$$

$\mu_X = P \circ X^{-1}$

induced me

$$E(X(\omega)) = \int_{\Omega} X(\omega)\, dP(\omega)$$

$$X(\omega) = \int_0^{X(\omega)} 1 \cdot d\mu(x)$$

$$\therefore E(X) = \int_{\Omega} \int_0^{X} d\mu\, dP$$

Mention the product space $(\Omega \times \mathbb{R}, \mathcal{A} \times \mathcal{B})$

mention the product meas. $\Phi = P \times \mu_X$ ≡ I think

$$= \int_{\Omega} \int_0^{\infty} I_{[0<t<X]}\, dt\, dP$$

Apply Tonelli / Fubini check the conditions required by the theorem for $(I_{[0<t<X]})$

$$= \int_0^{\infty} \int_{\Omega} I_{[0<t<X]}\, dP\, dt$$

$$= \int_0^{\infty} P(X>t)\, dt$$

$$= \int_0^{\infty} [1-F(t)]\, dt$$

Note: We cannot say density exists but we do know F can always be defined given $(\Omega, \mathcal{A}, P)$ and an X by $P(X^{-1}(-\infty, x]) = F(x)$

(b) $X \geq 0$ with df F

$$E(X^r) = \int_\Omega X^r \, dP$$

$$X^r = \int_0^X r t^{r-1} \, dt$$

$$\therefore E(X^r) = \int_\Omega \int_0^X r t^{r-1} \, dt \, dP = \int_\Omega \int_0^\infty r t^{r-1} I_{[0 \leq t \leq X]} \, dt \, dP$$

Mention the product space and product meas

⋮

use same arguements as before in (a)

(c) $E|X| < \infty$ given [Result holds also if only $E(X^+) < \infty$ OR $E(X^-) < \infty$]

Now $X = X^+ - X^-$

$$E(X) = E(X^+) - E(X^-)$$

Now $E(X) = \int_0^\infty (1 - F(t)) \, dt$ → to each $E(X^+)$ & $E(X^-)$

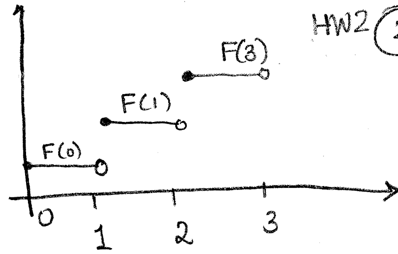
$X^+ = X$ if $X \geq 0$ $\qquad$ $X^- = -X$ if $X \leq 0$

0 ow $\qquad$ $= 0$ ow

$$\int_0^\infty P(X < -t) \, dt \longrightarrow -t = x \Rightarrow -dt = dx$$

HW2 2

#4] If $X \geq 0$ and integer valued.



$$E(X) = \int_0^\infty [1-F(x)]\,dx = \sum_{k=0}^{\infty} \int_{[k,k+1)} \underbrace{[1-F(x)]}\,dx$$

$$= \int_{[k,k+1)} [1-F(k)]\,dx = [1-F(k)](k+1-k)$$

$$= \sum_{k=0}^{\infty} 1-F(k)$$

$$= \sum_{k=0}^{\infty} P(X>k) = \sum_{k=0}^{\infty} P(X \geq k+1) = \sum_{t=1}^{\infty} P(X \geq t)$$

$$E(X^2) = \int_0^\infty 2t\,(1-F(t))\,dt \qquad \text{from } 3(b)$$

$$= \sum_{k=0}^{\infty} \int_{[k,k+1)} 2t\,(1-F(t))\,dt$$

$$= \sum_{k=0}^{\infty} (1-F(k)) \cdot \int_{[k,k+1)} 2t\,dt$$

$$= \sum_{k=0}^{\infty} [1-F(k)]\,[(k+1)^2 - k^2]$$

$$= \sum_{k=0}^{\infty} (2k+1)(1-F(k))$$

$$= \sum_{k=0}^{\infty} (2k+1)\,P(X>k)$$

$$X = \int_0^X 1 \, d\mu$$

$\mu \equiv$ counting meas on. $\mathbb{Z}$

$$= \int_{\{0\}} d\mu + \int_{(0,1)} d\mu + \int_{\{1\}} d\mu + \int_{(1,2)} d\mu + \cdots \int_{\{x\}} d\mu$$

$$= \underbrace{1 + 1 + \cdots 1}_{x \text{ times}}$$

$$E(X) = \int_{\Omega} \int_0^X d\mu(\omega) \, dP$$

$$= \int_{\Omega} \int_0^{\infty} I_{\{0 < \omega < X\}} \, d\mu(\omega) \, dP$$

define prod space and meas.

then use fubini's

$$E(X^r) = \int_0^{\infty} r t^{r-1} (1 - F(x)) \, dx$$

STA 5447 - Spring 2006

Homework 3 (and midterm preparation).

Assigned Th., Feb. the 9th.
Due Tue, Feb. the 21st.

- Ex 1. (A weak law of large numbers for uncorrelated summands).

Suppose that $X_1, X_2, \ldots$ are uncorrelated (i.e. $E(X_i X_j) = 0$ for any $i \neq j$) and that $E X_j^2 \leq M < \infty$. Show that:

1) $\bar{X} - E\bar{X} \xrightarrow[n \to \infty]{P} 0$.

2) $\bar{X} - E\bar{X} \xrightarrow[n \to \infty]{L_2} 0$,

where $\bar{X} \equiv \frac{1}{n} \sum_{i=1}^{n} X_i$.

WLLN : weaken conditions to uncorrelated with 2nd moment finite

Ex 2 Suppose that $X_1, X_2, \dots, X_n, \dots$ are independent and distributed as X with $P(X > x) = x^{-5}$, for all $x \geq 1$ and $n \geq 1$.

Show that: $\overline{\lim} \dfrac{\log X_n}{\log n} = c$ a.s. for some number c, and find c.

Ex 3 Show that if $\{X_n\}_{n\geq 1}$ is any sequence of r.v.s, there are constants $\{c_n\}_{n\geq 1}$ such that

$$\frac{X_n}{c_n} \xrightarrow[n\to\infty]{a.s.} 0 .$$

Ex 4. Let $X_1, \dots, X_n, \dots$ be independent r.v.s with $P(X_n = 1) = p_n$ and $P(X_n = 0) = 1 - p_n$.

Show that:

1) $X_n \xrightarrow{P} 0$ iff $p_n \xrightarrow[n\to\infty]{} 0$.

2) $X_n \xrightarrow{a.s.} 0$ iff $\sum_{n=1}^{\infty} p_n < \infty$.

Ex. 5. Suppose that $X_1, \ldots, X_n, \ldots$ are i.i.d. with $E|X_1| < \infty$. Show that:

$$\bar{X} \xrightarrow[n \to \infty]{L_1} \mu ;$$

where $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$; $\mu \equiv E X_1$.

Note that the SLLN always guarantees, under these hypotheses, that $\bar{X} \xrightarrow{a.s.} \mu$. This exercise shows that we also have L_1 convergence.

- Exercises 1 and 4: Shuva's recitation.
- Exercises 2, 3 and 5 presented in class.

[Not required]

1) $X_1, X_2 \ldots X_n \ldots$ are uncorrelated [ie $E[(X_i-\mu_i)(X_j-\mu_j)]=0 \quad \forall\, i\neq j$]

and $E(X_j^2) \leq M < \infty$

Show (a) $\overline{X} - E(\overline{X}) \xrightarrow{P} 0$

(b) $\overline{X} - E(\overline{X}) \xrightarrow{L_2} 0$

Sol^n: (b) To show $\overline{X} - E(\overline{X}) \xrightarrow{L_2} 0$

ie to show $E[(\overline{X} - E(\overline{X}))^2] \longrightarrow 0$

Consider $E[(\overline{X}-E(\overline{X}))^2] = E\left(\left[\frac{1}{n}\sum_{i=1}^{n}(X_i - E(X_i))\right]^2\right)$

$$= E\left\{\frac{1}{n^2}\sum_{i=1}^{n}(X_i-E(X_i))^2 + \frac{1}{n^2}\sum_{i\neq j}(X_i-E(X_i))(X_j-E(X_j))\right\}$$

$$= \frac{1}{n^2}\sum_{i=1}^{n} E[(X_i-E(X_i))^2] + \frac{1}{n^2}\sum_{i\neq j} E[(X_i-E(X_i))(X_j-E(X_j))]$$

$$= \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) + \frac{1}{n^2}(0) \qquad \{\because X_i\text{'s are uncorrelated}$$

$$\leq \frac{1}{n^2}\sum_{i=1}^{n} E(X_i^2)$$

$$\leq \frac{1}{n^2}\sum_{i=1}^{n} M$$

$$= \frac{M}{n}$$

$$\longrightarrow 0 \text{ as } n \longrightarrow \infty$$

$\therefore \overline{X} - E(\overline{X}) \xrightarrow{L_2} 0$

(a) To show $\bar{X} - E(\bar{X}) \xrightarrow{P} 0$

ie to show $P(|\bar{X} - E(\bar{X})| > \varepsilon) \longrightarrow 0$

$$P(|\bar{X} - E(\bar{X})| \geq \varepsilon) \leq \frac{Var(\bar{X})}{\varepsilon^2}$$

Now by (a) $Var(\bar{X}) = E[(\bar{X} - E(\bar{X}))^2] \longrightarrow 0$

$$\therefore \quad P(|\bar{X} - E(\bar{X})| \geq \varepsilon) \leq \frac{Var(\bar{X})}{\varepsilon^2} \longrightarrow 0$$

2] $X_1, X_2 \dots X_n \dots$ are indep and distributed as X with $P(X > x) = x^{-5}$ for $x \geq 1$, $n \geq 1$

Show $\overline{\lim} \dfrac{\log X_n}{\log n} = c$ a.s for some number 'c' and find c

<u>Sol^n</u>: We will show $\overline{\lim} \dfrac{\log X_n}{\log n} \geq c$

$$\overline{\lim} \frac{\log X_n}{\log n} \leq c$$

(i) $P(X_n > n^{c+\varepsilon}) = n^{-5(c+\varepsilon)}$ for any $\varepsilon > 0$

$$\sum_{n=1}^{\infty} P(\log X_n > (c+\varepsilon)\log n) = \sum_{n=1}^{\infty} \frac{1}{n^{5(c+\varepsilon)}}$$

$< \infty$ for any c st $5(c+\varepsilon) > 1$

We need $(c+\varepsilon) > \frac{1}{5}$ ie let $c = 1/5$

So we have $\sum_{n=1}^{\infty} P\left(\frac{\log X_n}{\log n} > c+\varepsilon\right) < \infty$

$$\Rightarrow \quad P\left(\left[\frac{\log X_n}{\log n} > c+\varepsilon\right] \text{ i.o}\right) = 0 \qquad \{\because \text{ Borel Cantelli 1}$$

$$\Rightarrow \quad P\left[\left(\overline{\lim} \frac{\log X_n}{\log n} > c+\varepsilon\right)\right] = 0 \qquad \left\{\begin{array}{l}\text{equivalent definition} \\ \text{of } \overline{\lim}\end{array}\right.$$

$$\Rightarrow \quad \overline{\lim_{n\to\infty}} \frac{\log X_n}{\log n} \leq c+\varepsilon \quad \text{a.s} \qquad \text{——} (*)$$

Now $P\left(X_n > n^{c-\varepsilon}\right) = n^{-5(c-\varepsilon)}$

$$\Rightarrow \sum_{n=1}^{\infty} P\left(\log X_n > (c-\varepsilon)\log n\right) = \sum_{n=1}^{\infty} \frac{1}{n^{5(c-\varepsilon)}}$$

$$= \infty \qquad \text{for any } c \text{ st } 5(c-\varepsilon) \leq 1$$

We need $c-\varepsilon \leq 1/5$ ie let $c = 1/5$ will do

So we have $\sum_{n=1}^{\infty} P\left(\frac{\log X_n}{\log n} > c-\varepsilon\right) = \infty$

$$\Rightarrow \quad P\left(\left[\frac{\log X_n}{\log n} > c-\varepsilon\right] \text{ i.o.}\right) = 1 \qquad \{\because \text{ Borel Cantelli 2}\}$$

ie $P\left(\overline{\lim} \dfrac{\log X_n}{\log n} \geq c-\varepsilon\right) = 1$

$\Rightarrow \quad \overline{\lim} \dfrac{\log X_n}{\log n} \geq c-\varepsilon \quad$ a.s $\quad$ —— (**)

From (*) & (**) taking $\varepsilon \to 0$ we get $\overline{\lim} \dfrac{\log X_n}{\log n} = c \equiv 1/5$

3] $\{X_n\}_{n\geq 1}$ is any sequence of r.v. Show $\exists$ constants c_n st $\dfrac{X_n}{c_n} \xrightarrow{a.s} 0$

Sol^n ~~Given any r.v $P(|X|>n) \to 0$ as $n \to \infty$~~

So we select $c_n \geq 0$ st $\quad P(n|X_n| > c_n) < 1/n^2$

For any arbitrary fixed $\varepsilon > 0$ $\exists$ N_0 st $\quad \varepsilon > 1/N_0$

$\therefore \quad P\left(\left|\dfrac{X_n}{c_n}\right| > \varepsilon\right) < P\left(\left|\dfrac{X_n}{c_n}\right| > \dfrac{1}{n}\right) \quad$ for all $n \geq N_0$

$$\therefore \quad \sum_{n=N_0}^{\infty} P\left(\left|\frac{X_n}{c_n}\right| > \varepsilon\right) < \sum_{n=N_0}^{\infty} P\left(\left|\frac{X_n}{c_n}\right| > \frac{1}{n}\right)$$

$$< \sum_{n=N_0}^{\infty} (1/n^2)$$

$$< \infty \qquad \left\{ \sum_{n=1}^{\infty} 1/n^2 < \infty \text{ so tail } < \infty \right.$$

$$\therefore \quad \sum_{n=1}^{\infty} P\left(\left|\frac{X_n}{c_n}\right| > \varepsilon\right) < \infty$$

$$\Rightarrow \quad P\left(\left|\frac{X_n}{C_n}\right| > \varepsilon \quad io \right) = 0$$

$$\Rightarrow \quad P\left(\overline{\lim} \left|\frac{X_n}{C_n}\right| > \varepsilon \right) = 0$$

$$\Rightarrow \quad \frac{X_n}{C_n} \xrightarrow{a.e} 0$$

4] $X_1, X_2 \cdots X_n \cdots$ are indep r.v with $P(X_n = 1) = p_n$

$P(X_n = 0) = 1 - p_n$

Show (1) $X_n \xrightarrow{P} 0$ iff $p_n \longrightarrow 0$

(2) $X_n \xrightarrow{a.s} 0$ iff $\sum_{n=1}^{\infty} p_n < \infty$

Sol^n (i) "$\Rightarrow$" Given $p_n \longrightarrow 0$ to show $X_n \xrightarrow{P} 0$

ie to show $P(|X_n| > \varepsilon) \longrightarrow 0$

$$P(|X_n| > \varepsilon) = P(|X_n| = 1) \qquad \because \{\omega \mid |X_n| > \varepsilon\} = \{\omega \mid X_n = 1\}$$

when $0 < \varepsilon \leq 1$

$$= P(X_n = 1)$$

$$= p_n \qquad \{ \text{For } \varepsilon > 1 \quad P(|X_n| > \varepsilon) = P(\phi) = 0$$

$$\lim_{n \to \infty} P(|X_n| > \varepsilon) = \lim_{n \to \infty} p_n = 0$$

$$\therefore \quad X_n \xrightarrow{P} 0$$

" $\Rightarrow$ " given $X_n \xrightarrow{P} 0$ to show $p_n \longrightarrow 0$

$$P(|X_n| > \varepsilon) = P(X_n = 1) = p_n$$

$$\lim_{n\to\infty} p_n = \lim_{n\to\infty} P(|X_n| > \varepsilon)$$

$$= 0$$

u $p_n \longrightarrow 0$

(2) " $\Leftarrow$ " $\sum_{n=1}^{\infty} p_n < \infty$ (given)

$$\Leftrightarrow \sum_{n=1}^{\infty} P(|X_n| > \varepsilon) < \infty$$

$$\Rightarrow P(|X_n| > \varepsilon \ io) = 0$$ { Borell Cantelli

$$\Rightarrow P(\overline{\lim} \ |X_n| > \varepsilon) = 0$$

$$\Rightarrow X_n \xrightarrow{a.s} 0$$

" $\Rightarrow$ " $X_n \xrightarrow{a.s} 0$ (given) to show $\sum_{n=1}^{\infty} p_n < \infty$

We will show $\sum_{n=1}^{\infty} p_n = \infty \Rightarrow X_n \overset{a.s.}{\not\to} 0$

Now $\sum_{n=1}^{\infty} p_n = \infty \Rightarrow \sum_{n=1}^{\infty} P(|X_n| > \varepsilon) = \infty$

$$\Rightarrow P(|X_n| > \varepsilon \ io) = 1$$ { Borel Cantelli

$$\Rightarrow X_n \overset{a.s}{\not\to} 0$$

5] $X_1, X_2 \dots X_n \dots$ are iid with $E|X_1| < \infty$ with $\mu \equiv E(X_1)$

Show $\bar{X} \xrightarrow{\mathcal{L}_1} \mu$

Sol^n: $\{X_n\}_{n\geq 1}$ are iid r.v with $E|X_1| < \infty$

$\therefore \quad \bar{X}_n \xrightarrow{a.s} \mu \qquad$ [by SLLN]

$\Rightarrow \quad \bar{X}_n \xrightarrow{P} \mu \qquad [\because P(\Omega) < \infty \quad \xrightarrow{a.s} \Rightarrow \xrightarrow{P}]$

Since $E|X_1| < \infty \Rightarrow |X_n| \in \mathcal{L}_1 \quad \forall n$

Now $E|\bar{X}_n| \leq \frac{1}{n} \sum_{i=1}^{n} E|X_i| = E(X_1) < \infty$

$\Rightarrow \quad \bar{X}_n \in \mathcal{L}_1$

So if we can show $\{|\bar{X}_n| : n \geq 1\}$ are u.i then we can use Vitali's $\Rightarrow \bar{X}_n \xrightarrow{\mathcal{L}_1} \mu$

- To show u.i of $\bar{X}_n = \sum_{i=1}^{n} \frac{X_i}{n}$ we will show $\left\{\frac{1}{n} \sum_{i=1}^{n} |X_i|\right\}$ is u.i

Now X_i's are iid $\Rightarrow |X_i|$ are iid

$E|X_1| < \infty \Rightarrow \frac{\sum_{i=1}^{n} |X_i|}{n} \xrightarrow{a.s} E|X_1| \qquad$ {by SLLN

$\Rightarrow \frac{\sum_{i=1}^{n} |X_i|}{n} \xrightarrow{P} E|X_1|$

Also $E\left(\frac{\sum_{i=1}^{n} |X_i|}{n}\right) = \frac{1}{n} \sum_{i=1}^{n} E|X_i| = \frac{1}{n} \sum_{i=1}^{n} E|X_1| = E|X_1|$

$\therefore \quad \overline{\lim} \; E\left(\dfrac{\sum_{i=1}^{n}|X_i|}{n}\right) = \overline{\lim} \; E|X_1|$

$= E|X_1| < \infty$

So by Vitali's $\left\{ \dfrac{\sum_{i=1}^{n}|X_i|}{n} \right\}$ are u.i $\Rightarrow \begin{cases} ① \; \sup_n E\left(\dfrac{\sum |X_i|}{n}\right) < \infty \\ ② \; \sup_n \int_A \dfrac{\sum |X_i|}{n} \, d\mu < \varepsilon \text{ for any } A \text{ st } \mu(A) < \delta \end{cases}$

Now $\left| \dfrac{\sum_{i=1}^{n} X_i}{n} \right| \leq \dfrac{\sum_{i=1}^{n} |X_i|}{n}$.

$\therefore \quad E\left|\dfrac{\sum X_i}{n}\right| \leq E\left(\dfrac{\sum |X_i|}{n}\right)$

$\therefore \quad \sup_n E\left(\left|\dfrac{\sum_{i=1}^{n} X_i}{n}\right|\right) \leq \sup_n E\left(\dfrac{\sum_{i=1}^{n}|X_i|}{n}\right) < \infty$ {from ①

Also for any A st $\mu(A) < \delta$

$\sup_n \int_A \left| \dfrac{\sum_{i=1}^{n} X_i}{n} \right| d\mu \leq \sup \int_A \dfrac{\sum_{i=1}^{n} |X_i|}{n} d\mu < \varepsilon$ {from ②

$\therefore \quad \{ |\bar{X}_n| \}$ are u.i

$\{ |\bar{X}_n| : n \geq 1 \}$ are u.i — qed.

$\int_{[|\bar{X}_n| \geq \lambda]} |\bar{X}_n| \, d\mu \longrightarrow 0$ as $\lambda \to \infty$ {from def of unif integrability}

$$F_n(d_n) \equiv P(|X_n| \le d_n)$$

Let $d_n = F_n^{-1}\left(1-\frac{1}{n^2}\right)$

$$P(|X_n| \le d_n) = F_n\left(F_n^{-1}(1-1/n^2)\right)$$

$$\ge \left(1-\frac{1}{n^2}\right)$$

$$\therefore \quad P(|X_n| > d_n) \le \frac{1}{n^2}$$

$$\therefore \quad \sum_{n=1}^{\infty} P(|X_n| > d_n) \le \sum_{n=1}^{\infty} 1/n^2 < \infty$$

$$\therefore \quad \sum_{n=1}^{\infty} P\left(\left|\frac{X_n}{d_n}\right| > 1\right) < \infty$$

$$\Rightarrow \quad P\left[\left(\left|\frac{X_n}{d_n}\right| > 1\right) \text{ i.o.}\right] = 0$$

$$\therefore \quad \left|\frac{X_n}{d_n}\right| \le 1 \quad \text{a.s}$$

Let $C_n = n d_n$

$$\left|\frac{X_n}{d_n}\right| = \left|\frac{X_n}{n d_n}\right| = \left|\frac{(X_n/d_n)}{n}\right|$$

Since $\left(\frac{X_n}{d_n}\right)$ is bdd a.s $\Rightarrow \frac{(X_n/d_n)}{n} \longrightarrow 0$ as $n \to \infty$

Fix X $n \to \infty$.

$$P(|X|>n)$$

$$= P(X>n) + P(X<-n).$$

$$\leq \boxed{1-P(X\leq n)} + \boxed{P(X\leq -n)}$$

$$\to 0 \qquad \to 0$$

$$\boxed{\lim_{x\to-\infty} F(x) = 0}$$

$$\boxed{\lim_{x\to\infty} F(x) = 1}$$

$$\therefore P(|X|>n) \leq 0$$

$$P(|X|>n) \leq \frac{E|X|}{n}$$

$$X_n(\omega) = n+1 \quad \forall \omega$$

$$P(|X_n|>n) = 1 \quad \forall n$$

#3 $F_n(d_n) = P(|X_n| \le d_n)$

$d_n \equiv F^{-1}(1 - 1/n^2)$

$$P(|X_n| \le d_n) = F_n(d_n)$$

$$= F_n\left(F^{-1}(1 - 1/n^2)\right)$$

$$\ge \left(1 - \frac{1}{n^2}\right)$$

$$P(|X_n| > d_n) \le \frac{1}{n^2}$$

$$\sum_{n=1}^{\infty} P(|X_n| > d_n) \le \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

$$\Rightarrow P\left(\left|\frac{X_n}{d_n}\right| > 1 \text{ i.o}\right) = 0$$

$$\Rightarrow \left|\frac{X_n}{d_n}\right| \le 1 \quad \text{a.s}$$

$$\frac{\left(\frac{X_n}{d_n}\right)}{n} \qquad \frac{X_n}{C_n} = \frac{X_n}{n d_n}$$

Let $C_n = n d_n$

$$\Rightarrow \frac{\left(\frac{X_n}{d_n}\right)}{n} \longrightarrow 0 \quad \text{a.s} \quad \left\{ \text{since } \left|\frac{X_n}{d_n}\right| \text{ is bdd} \right.$$

Rescaled variable $\longrightarrow 0$ Inappropriate scaling

Theorems : Def Proofs Statements

Encercises - 3 $\rightarrow$ one will be something from proofs.

$\rightarrow$ one will be similar to HW.

$\rightarrow$ one will be something new.

To show $\qquad U_n \xrightarrow{a.s} 0$

$$\Longleftrightarrow \quad P(|U_n| > \varepsilon \ i.o) = 0$$

reduces to showing

$$\sum_{n=1}^{\infty} P(|U_n| > \varepsilon) < \infty$$

Ex 4] Let $X_1 \dots X_n \dots$ be indep r.v.

$$P(X_n = 1) = p_n \qquad P(X_n = 0) = 1 - p_n.$$

Show (i) $X_n \xrightarrow{P} 0 \iff p_n \to 0$

(ii) $X_n \xrightarrow{a.s} 0 \iff \sum_{n=1}^{\infty} p_n < \infty$

Recall

Now for indep r.v the Borel cantelli lemma says.

$$\sum P(A_n) < \infty \iff P(A_n \text{ i.o}) = 0.$$

Not needed here.

Proof (i) "$\Rightarrow$"

Given $X_n \xrightarrow{P} 0$

$\Rightarrow$. $\forall\ \varepsilon > 0$ $\lim P(|X_n| > \varepsilon) = 0$

Now $\{\omega : |X_n| > \varepsilon\} = \{\omega : X_n = 1\}$ $[\because X_n = 0 \text{ or } 1$

$\therefore$ $\lim_{n\to\infty} P(X_n = 1) = 0.$

ie $\lim_{n\to\infty} p_n = 0$

ie $p_n \longrightarrow 0$

"$\Leftarrow$"

Given $p_n \longrightarrow 0$

ie $\lim_{n\to\infty} P(X_n = 1) = 0$

$\Rightarrow$ $\lim_{n\to\infty} P(|X_n| > \varepsilon) = 0$ $\forall\ \varepsilon > 0$ $\{\because \{X_n = 1\} = \{|X_n| > \varepsilon\}$

$\Rightarrow$ $X_n \xrightarrow{P} 0$

(ii) "$\Leftarrow$" given $\sum_{n=1}^{\infty} p_n < \infty$

$$\Leftrightarrow \quad \sum_{n=1}^{\infty} P(X_n = 1) < \infty$$

$$\Leftrightarrow \quad \sum_{n=1}^{\infty} P(|X_n| > \varepsilon) < \infty \quad \text{for any } \varepsilon > 0$$

$$\Rightarrow \quad P(|X_n| > \varepsilon \text{ i.o}) = 0 \quad \left\{ \text{by Borel Cantelli} \atop A_n = \{\omega : |X_n(\omega)| > \varepsilon\} \right.$$

Now $\{\omega : X_n(\omega) \to X(\omega)\}$ ie $X_n \xrightarrow{a.s} X$

$$\Leftrightarrow \quad \forall\, \varepsilon > 0 \quad \exists\, n \text{ st } P\left(\bigcup_{n=1}^{\infty} \bigcap_{m \geq n}^{\infty} \{\omega : |X_m(\omega) - X(\omega)| < \varepsilon\} \right) \to 1$$

$$\Leftrightarrow \text{ ie } \quad \forall\, \varepsilon > 0 \quad \exists\, n \text{ st } P\left[\bigcap_{n=1}^{\infty} \bigcup_{m \geq n}^{\infty} \underbrace{\{|X_m - X| > \varepsilon\}}_{B_m} \right] \to 0$$

$$\Leftrightarrow \quad \exists\, n \text{ st } P(\overline{\lim}\ B_m) \to 0$$

$$\Leftrightarrow \quad \text{st } P(B_m \text{ i.o}) \to 0$$

We have $P(\lvert X_n\rvert > \varepsilon \text{ i.o}) = 0$ $\Rightarrow \quad X_n \xrightarrow{a.s} X$

If $P(A_n) = 1$ then $P\left(\bigcap_{n=1}^{\infty} A_n\right) = 1$.

$$P\left(\bigcup_{n=1}^{\infty} A_n^c\right) \leq \sum_{n=1}^{\infty} P(A_n^c) = 0.$$

$$\therefore \quad P\left(\bigcap_{n=1}^{\infty} A_n\right) = 1 - P\left(\bigcup_{n=1}^{\infty} A_n^c\right) = 1 - 0 = 1$$

"$\Rightarrow$" given $X_n \xrightarrow{a.s} 0$ To show $\sum_{n=1}^{\infty} p_n < \infty$.

We will prove $\left(\sum_{n=1}^{\infty} p_n \right) = \infty \Rightarrow X_n \not\xrightarrow{a.s} 0$

Now $\sum_{n=1}^{\infty} p_n = \infty \Rightarrow \sum_{n=1}^{\infty} P(X_n = 1) = \infty$

$\Leftrightarrow \sum_{n=1}^{\infty} P(|X_n| > \varepsilon) = \infty$

$\Leftrightarrow P(|X_n| > \varepsilon \text{ io}) = 1$ $\left\{ \begin{array}{l} \because X_n\text{'s are inde}\\ \text{Borel cantelli} \end{array} \right.$

$\Rightarrow X_n \not\xrightarrow{a.s} 0$ [equivalent definitions of $\xrightarrow{a.s}$

$\boxed{Ex\ 1}$: X_i are uncorrelated ie $E(X_i X_j) = 0$ and $E(X_i^2) \leq M < \infty$

[they are not indep or identically distributed]

Step 1: Show $\bar{X} - E(\bar{X}) \xrightarrow{L_2} 0$ ie $E[(\bar{X} - E(\bar{X}))^2] \longrightarrow 0$

Step 2: Show $\bar{X} - E(\bar{X}) \xrightarrow{P} 0$ ie $P(|\bar{X} - E(\bar{X})| > \varepsilon) \longrightarrow 0$

Using Chebyshev & step 1 $P(|\bar{X} - E(\bar{X})| \geq \varepsilon) \leq \dfrac{Var(\bar{X})}{\varepsilon^2}$

Proof: Step 1

$$E(\bar{X} - E(\bar{X}))^2 = E\left[\frac{1}{n}\sum_{i=1}^{n}(X_i - E(X_i))\right]^2$$

$$= E\left\{ \frac{1}{n^2} \sum (X_i - E(X_i))^2 + \frac{2}{n} \sum_{i<j} (X_i - E(X_i))(X_j - E(X_j)) \right.$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} \left[E(X_i^2) - E(X_i)^2 \right] \qquad \{\text{uncorrelated}\}$$

$$\leq \frac{1}{n^2} \sum_{i=1}^{n} E(X_i^2)$$

$$\leq \frac{nM}{n^2}$$

$$= M/n$$

$$\longrightarrow 0 \qquad \text{as} \quad n \longrightarrow \infty$$

#5

X_i are iid $\Rightarrow$ $|X_i|$ are iid

$E|X_1| < \infty$

By SLLN $\quad \frac{1}{n}\sum_{i=1}^{n} |X_i| \xrightarrow{a.s} E|X_1|$

$$\frac{1}{n}\sum_{i=1}^{n} |X_i| \xrightarrow{P} E|X_1|.$$

$$\overline{\lim}\ E\left(\frac{1}{n}\sum_{i=1}^{n}|X_i|\right) = \overline{\lim}\left(\frac{1}{n}\sum_{i=1}^{n} E|X_i|\right)$$

$$= \overline{\lim}\left(\frac{1}{n}\sum (E|X_1|)\right)$$

$$= \overline{\lim}\ E|X_1|$$

$$= E|X_1|$$

So we have
$$\overline{\lim}\ E\left(\frac{1}{n}\sum_{i=1}^{n}|X_i|\right) = E|X_1| < \infty$$

$\Rightarrow$ $\left\{\frac{\sum_{i=1}^{n}|X_i|}{n}\text{'s are u.i}\right\}$

$\Rightarrow$ $\left\{\frac{\sum X_i}{n}\right\}$ are u.i $\leftarrow$ (Prove)

$$\left|\frac{\sum X_i}{n}\right| \le \frac{\sum |X_i|}{n} \Rightarrow \sup E|\bar{X}_n| \le \sup \left|\frac{\sum |X_i|}{n}\right| < \infty$$

$$\sup_A \int |\bar{X}_n|\, d\mu \le \sup_A \int \frac{\sum |X_i|}{n}\, d\mu \searrow 0$$

```
. list

     +-------------------------------------------------------------------+
     | id   randate   lastfu              treat       wt24    lengthfu |
     |-------------------------------------------------------------------|
  1. |  1     15631    16195   Active Treatment          .    1.544148 |
  2. |  2     16268    16449            Placebo          .     .495551 |
  3. |  3     16440        .   Active Treatment          .           . |
  4. |  4     16083    16468   Active Treatment          .    1.054072 |
  5. |  5     15953        .   Active Treatment   73.00000           . |
     |-------------------------------------------------------------------|
  6. |  6     15987        .   Active Treatment          .           . |
  7. |  7     16496        .            Placebo          .           . |
  8. |  8     16566        .   Active Treatment          .           . |
  9. |  9     15558    15679   Active Treatment          .    .3312799 |
 10. | 10     16134    16497            Placebo          .    .9938399 |
     |-------------------------------------------------------------------|
 11. | 11     16314        .   Active Treatment          .           . |
 12. | 12     15944        .   Active Treatment   76.00000           . |
 13. | 13     16212        .   Active Treatment          .           . |
 14. | 14     15595    16272            Placebo          .    1.853525 |
 15. | 15     16153        .            Placebo          .           . |
     |-------------------------------------------------------------------|
 16. | 16     16184        .            Placebo          .           . |
 17. | 17     15372    15550   Active Treatment          .    .4873374 |
 18. | 18     15400    15572            Placebo          .    .4709103 |
 19. | 19     15897        .            Placebo   65.00000           . |
 20. | 20     15918    16481   Active Treatment   61.00000     1.54141 |
     |-------------------------------------------------------------------|
 21. | 21     16575        .   Active Treatment          .           . |
 22. | 22     15790    16226   Active Treatment          .    1.193703 |
 23. | 23     16568        .   Active Treatment          .           . |
 24. | 24     15603    15835   Active Treatment          .    .6351814 |
```

2674419807

If X_i's are iid $\in L_1$

$\Downarrow$

X_i's are u.i

OR

$|X_i|$'s are u.i

$$\sup_n E|X_n| \mathbb{1}_{\{|X_n| \geq \lambda\}} = E|X_1| \cdot \mathbb{1}_{\{|X_1| \geq \lambda\}} \xrightarrow{} 0 \text{ as } \lambda \to \infty.$$

$|\overline{X}_n|$ are ui as well

$$X_n = \frac{1}{n}\sum_{i=1}^{n}|X_i| \xrightarrow[p]{} E|X|$$

X_i's are iid

$\underline{|X_i|\text{'s are iid}}$

$\Rightarrow$

$\Rightarrow \frac{1}{n}\sum_{i=1}^{n}|X_i| \xrightarrow[p]{} E|X_1|$

$\underline{\text{Tarry}}$

$$\overline{\lim}\ E\left(\frac{1}{n}\sum_{i=1}^{n}|X_i|\right) = E|X_1|$$

(1a) $\frac{1}{n}\sum_{i=1}^{n}|X_i|$ $\qquad \left|\frac{1}{n}\sum_{i=1}^{n}X_i\right| \le \frac{1}{n}\sum_{i=1}^{n}|X_i|$

$\frac{1}{n}\sum_{i=1}^{n}|X_i|$ is $\underline{ui}$.

$\Rightarrow \frac{1}{n}\sum_{i=1}^{n}X_i$ is $\underline{ui}$

1

~~$P\ X_n >$~~

$$P(|X_n| > \varepsilon \ i.o) = 0 \iff a.s.$$

$$\updownarrow$$

$$\sum P(|X_n| > \varepsilon) < \infty.$$

$$Y_n = \frac{X_n}{c_n} \qquad P(|X| > c) \xrightarrow[c \to \infty]{} 0.$$

$$P\left(\frac{|X_n|}{c_n} > \varepsilon\right) < 1/n^2$$

$$P(|X_n| > c_n \varepsilon)$$

① $\varepsilon > 0$ arbit

② $\forall n, \exists c_n$

$P(|X_n| > c_n \varepsilon) < 1/n^2$

$$\sum P\left(\frac{|X_n|}{c} > \varepsilon\right) < \sum \frac{1}{n^2}$$

$$\sum P\left(\frac{|X_n|}{c_n} > \varepsilon\right) < \infty$$

$$\Downarrow$$

$$P\left(\left|\frac{X_n}{c_n}\right| > \varepsilon \ i.o.\right) = 0$$

Homework 4

Assigned. March the 2nd 2006
Due March the 14th 2006.

Use two series theorem. lecture 24

Exercise 1. Let $X_K \sim$ Uniform $[-K, K]$, for $K \geq 1$.
Let a be a fixed constant, $0 < a < 1$.

1) Show that $\sum_{K=1}^{n} a^K X_K \xrightarrow{a.s.}$ (some r.v.) S.

2) Evaluate ES and $\text{var}\, S$.

Exercise 2 Let $X_1, \dots, X_n$ be a sample of size n.
Sampling with replacement from $\{X_1, \dots, X_n\} = A$ means drawing one value from A, putting it back and continuing the process.

1) What discrete random variable Y summarizes this process (sampling with replacement)?

2) Show that sampling from F_Y (the cdf of Y) is equivalent to sampling from F_n, where F_n is the empirical distribution function associated with the sample.

Y is a discrete r.v taking values $(x_1 \ldots x_n)$

$$P(Y = x_j) = \frac{1}{n}$$

$$F_Y(x) = P(Y \le x) = \sum_{\{x_j \le x\}} \left(\frac{1}{n}\right) = \frac{1}{n} \{\# \text{ of } x_j \le x\}$$

$$= \frac{1}{n} \sum_{j=1}^{n} I_{[x_j \le x]}$$

$$= F_n(x).$$

Exercise 3.

Let $h : [0,1] \to [0,1]$ be continuous.

Let $\xi^{(n)} \equiv (\xi_1, \dots, \xi_n)$ be iid $\mathrm{Unif}(0,1)$.

Let $Z^{(n)} \equiv (Z_1, \dots, Z_n)$ be iid $\mathrm{Unif}(0,1)$.

Assume $\xi^{(n)}$ is independent of $Z^{(n)}$.

We showed (Lecture 23) that

$$\overline{X}_n \equiv \frac{1}{n} \sum_{k=1}^{n} 1[h(\xi_k) \geq Z_k] \xrightarrow{a.s.} \int_0^1 h(s)\,ds.$$

1) Show that $\overline{Y}_n \equiv \frac{1}{n} \sum_{k=1}^{n} h(\xi_k) \xrightarrow{a.s.} \int_0^1 h(s)\,ds.$

2) Calculate $\mathrm{var}(\overline{X}_n)$ and $\mathrm{var}(\overline{Y}_n)$. Based on this calculation, what estimator would you prefer?

3) Design a simulation study to investigate whether your findings from 2) and 1) can be confirmed in small samples.

Simulation plan

1) Consider 3 functions h (more if you'd like). Take 2 continuous and 1 with some discontinuities (a step function, for instance).

2) For **each** function h consider (say) 3 sample sizes: $n = 50$, $n = 200$, $n = 500$.

3) For **each** combination (h, n) do the following:

(*) $\begin{cases} \text{Compute } \bar{X}_n. \text{ Save it.} \\ \text{Compute } \left[\bar{X}_n - \int_0^1 h(x)dx\right]^2. \text{ Save it.} \end{cases}$

(**) $\begin{cases} \text{Compute } \bar{Y}_n. \text{ Save it.} \\ \text{Compute } \left[\bar{Y}_n - \int_0^1 h(x)dx\right]^2. \text{ Save it.} \end{cases}$

Repeat (*) $r = 100$ (say) times to obtain $\bar{X}_{n,k}$, $\left[\bar{X}_{n,k} - \int_0^1 h(x)dx\right]^2$; $k = 1, \dots, 100$.

Compute $\frac{1}{100}\sum_{k=1}^{100} \bar{X}_{n,k}$ and $\frac{1}{100}\sum_{k=1}^{100}\left[\bar{X}_{n,k} - \int_0^1 h(x)dx\right]^2$.

Do the same for (**).

- Present then your results in a table.

Please provide an estimate of var($\bar{X}_n$) and var($\bar{Y}_n$). You have complete freedom here.

$$\frac{1}{n} \sum_{k=1}^{n} h(\xi_k) \xrightarrow{a.s} \int_0^1 h(x)\,dx$$

Make groups to do the simulation study

Homework 4

Exercise 1:

Let $X_k \sim Uniform[-k,k], for\, k \geq 1$.

Let a be a fixed constant such that $0 < a < 1$

1) Show that $\sum_{k=1}^{n} a^k X_k \to a.s.\ S$ where S is some random variable

Note: In order to show (1) we will use part one of Theorem 2 (The Two Series Theorem) which is located in *Convergence of a Series of Independent Random Variables* section of Lecture 24.

The Two Series Theorem states: Let $X_1, \ldots, X_n$ be independent with $X_k \cong (\mu_k, \sigma_k)$

I) If $\sum_{k=1}^{n} \mu_k \to \mu$ and $\sum_{k=1}^{\infty} \sigma_k^2 < \infty$, then $S_n \equiv \sum_{k=1}^{n} X_k \to a.s. S$ where S is a r.v.

Clearly, $E(X_k) = 0$ *for all k*, hence $\mu_k = E(a^k X_k) = a^k E(X_k) = 0$ *for all k*

Therefore, $(a^k X_k) \cong (0, \sigma_k)$ This implies that $\sum_{k=1}^{n} \mu_k \to \mu$

Note: the will be completed by showing that $\sum_{k=1}^{\infty} \sigma_k^2 < \infty$

$$Var(a^k X_k) = E(a^k X_k)^2 - (E(a^k X_k))^2 = E(a^k X_k)^2 = a^{2k} E(X_k)^2 = a^{2k}\left(\frac{k^2}{3}\right) = \frac{1}{3}((a^k)k)^2$$

Note: We want show to that $\forall a \in (0,1)$, $\exists N$ such that $\forall k > N$ it follows that $(a^k)k < \frac{1}{k}$

Facts: $\sum_{k=1}^{\infty} \left(\frac{1}{k}\right)^2 < \infty$ and for $n \geq 3$ $\frac{1}{n}\log_b(n) \to 0$ monotonically from above

Let $b = \frac{1}{a}$ then if we choose N such that $\frac{1}{N}\log_b N < \frac{1}{2}$, then $\forall n > N$ we have

$$2\log_b n < n \Leftrightarrow 2\log_b n < n\log_b b \Leftrightarrow \log_b n^2 < \log_b b^n \Leftrightarrow n^2 < b^n \Leftrightarrow n < nb^n \Leftrightarrow \frac{1}{n} > \frac{n}{b^n} = (a^n)n$$

Therefore, $\frac{1}{3}\sum_{k=N+1}^{\infty} ((a^k)k)^2 < \sum_{k=1}^{\infty}\left(\frac{1}{k}\right)^2 < \infty$ which implies that $\sum_{k=1}^{\infty} \sigma_k^2 = \frac{1}{3}\sum_{k=1}^{\infty}((a^k)k)^2 < \infty$

Therefore, by The Two Series Theorem it follows that

$S_n \equiv \sum_{k=1}^{n} X_k \to a.s. S$ where S is a r.v. Q.E.D.

$E(S) = 0$

$$V(S) = \frac{1}{9} \frac{a^2(a^2+1)}{(1-a^2)^3}$$

We need $\sum_{k=1}^{\infty} \sigma_k^2 < \infty$

ie $\sum_{k=1}^{\infty} a_k^2 \frac{k^2}{3} < \infty$

ie to show $\sum_{k=1}^{\infty} \left(a_k \frac{k}{\sqrt{3}}\right)^2 < \infty$

If we show $a_k \frac{k}{\sqrt{3}} < \frac{1}{k}$ for $k > N$

$$a_k k^2 < \sqrt{3} \qquad \text{for } k > N$$

$$a_k < \frac{\sqrt{3}}{k^2}$$

If $a_k = \left(\frac{1}{2}\right)^k$ then $k^2 \leq 2^k \sqrt{3} \quad \forall k \geq N$

By 2 series theorem $E(S) = \sum_{k=1}^{\infty} \mu_k = 0$

$$Var(S) = \sum_{n=1}^{\infty} \frac{a^{2n} n^2}{3}$$

Now for $a \in [0,1]$ then $\sum_{n=0}^{\infty} a^{2n} = \frac{1}{1-a^2}$

$$\sum_{n=0}^{\infty} 2n\, a^{2n-1} = \frac{d}{da}\left(\frac{1}{1-a^2}\right) \equiv d_0$$

$$\sum_{n=0}^{\infty} 2n(2n-1) a^{2n-2} = \frac{d}{da}\left(\frac{1}{1-a^2}\right) \equiv d_1$$

$\sum_{n=0}^{\infty} (4n^2 - 2n) a^{2n-2} = d_1$

$\Updownarrow$

$\frac{4}{a^2} \sum_{k=1}^{\infty} k^2 a^{2k} - \frac{2}{a^2} \sum_{k=1}^{\infty} k a^{2k} = d_1$

$\frac{4}{a^2} \sum k^2 a^{2k} - \frac{2}{a^2}\left(\frac{a d_0}{2}\right)$

$V(S) = \left[\left(\frac{1}{1-a^2}\right) + \frac{1}{a}\left(\frac{1}{1-a^2}\right)\right] \frac{a^2}{12}$

H.W #4.

Q2(a) Let Y be discrete Random Variable which takes following values.

$$Y = \{X_1, X_2, \ldots, X_n\}$$

$$P(Y=t) = \begin{cases} 1/n, & t = X_1, X_2, \ldots, X_n \\ 0, & \text{otherwise} \end{cases}$$

$\therefore$ Y is a uniform discrete R.V.

(b) $$F_n(t) = \frac{1}{n}\sum_{i=1}^{n} 1_{[X_i \le t]}$$

$$F_Y(x) = P(Y \le x) = \sum_{\{X_j \le x\}} \frac{1}{n} = \frac{\#\text{ of } \{X_j : X_j \le x\}}{n} = \frac{\sum_{j=1}^{n} 1_{[X_j \le x]}}{n}$$

$$= \frac{1}{n}\sum_{j=1}^{n} 1_{[X_j \le x]}.$$

Hence, Sampling from $F_n(x)$ is same as sampling from $F_Y(x)$.

Q3 (a). $Y_n = h(\xi_n)$, $Y_1, Y_2, \ldots, Y_n$ iid. Y.

and $E(Y_n) = E(h(\xi_n)) < \infty$

Using SLLN.

$$\overline{Y}_n \xrightarrow[a.s.]{} E(Y_1)$$

$$E(Y_1) = E(h(\xi_1)) = E(h(\xi)) = \int_0^1 h(s)\, ds.$$

(b) (i) $Var(\overline{X}_n) = ?$

$$Var(\overline{X}_n) = Var\left(\frac{X_1 + X_2 + \cdots + X_n}{n}\right) = \frac{n}{n^2} Var(X_1)$$

$$= \frac{1}{n}\left[EX^2 - E^2X\right]$$

$$E(X) = E\left[1_{[h(\xi)\geq Z]}\right] = \int_0^1\int_0^1 1_{[h(t)\geq s]}\, ds\, dt = \int_0^1 \left(\int_0^{h(t)} ds\right) dt$$

$$= \int_0^1 h(t)\, dt.$$

$$E(X^2) = E\left[1_{[h(\xi)\geq Z]} \cdot 1_{[h(\xi)\geq Z]}\right] = E\left[1_{[h(\xi)\geq Z]}\right]$$

$$= \int_0^1 h(t)\, dt.$$

$$\therefore Var(\overline{X}_n) = \frac{1}{n}\left[\int_0^1 h(t)\,dt - \left[\int_0^1 h(t)\,dt\right]^2\right]$$

(ii) $Var(\bar{Y}_n) = ?$

$Y_n = h(\xi_n)$

$Var(\bar{Y}_n) = \frac{1}{n}(EY^2 - E^2Y)$ {Using the same argument we used for $\bar{X}_n$.}

$$E(Y_n) = E(h(\xi_n)) = E(h(\xi)) = \int_0^1 h(t)\,dt = E(Y)$$

$$E(Y^2) = E(h^2(\xi)) = \int_0^1 h^2(t)\,dt.$$

$$\therefore \quad Var(\bar{Y}_n) = \frac{1}{n}\left[\int_0^1 h^2(t)\,dt - \left(\int_0^1 h(t)\,dt\right)^2\right].$$

and $Var(\bar{X}_n) = \frac{1}{n}\left[\int_0^1 h(t)\,dt - \left[\int_0^1 h(t)\,dt\right]^2\right].$

Since $h: [0,1] \longrightarrow [0,1]$

$\therefore \quad h^2(t) \leq h(t)$

Hence $\int_0^1 h^2(t)\,dt \leq \int_0^1 h(t)\,dt$

$\therefore \quad Var(\bar{Y}_n) \leq Var(\bar{X}_n)$

Homework 5 – STA 5447.

Due Tuesday, March the 28^{th} 2006.

Exercise 1

Prove (100), (101) and (102) page 10, Lecture 25.

Exercise 2

Let (X_i, Y_i) be iid with $\left[\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1 & \rho\\ \rho & 1\end{pmatrix}\right]$,

for $1 \le i \le n$.

Let $\hat{\rho}_n \equiv \dfrac{S_{XY}}{S_X \cdot S_Y}$, where

$$S_{XY} \equiv \sum_{i=1}^{n} (X_i - \bar{X}_n)(Y_i - \bar{Y}_n),$$

$$S_X \equiv \left[\sum_{i=1}^{n} (X_i - \bar{X}_n)^2\right]^{1/2},$$

$$S_Y \equiv \left[\sum_{i=1}^{n} (Y_i - \bar{Y}_n)^2\right]^{1/2}.$$

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1) Show that $\sqrt{n}(\hat{\rho}_n - \rho) \xrightarrow{d} N(0, \sigma^2)$.

2) Evaluate σ^2.

(Hint: Use the δ-method.)

#1] $X_{nk} \simeq F_{nk}(\mu_{nk}, \sigma_{nk}^2)$

Define $Y_k \equiv \dfrac{X_{nk} - \mu_{nk}}{S_n}$

$$S_n = \left\{ \sum_{k=1}^{n} \sigma_{nk}^2 \right\}^{1/2}$$

$Y_k' \equiv Y_k I_{[|Y_k| \le \varepsilon_n]}$

$\mu_k^* = E(Y_k')$

$\sigma_k^{2*} = Var(Y_k')$

Show

(100) : $\sum_{k=1}^{n} \mu_k^* \xrightarrow[n\to\infty]{} 0$

(101) $\sum_{k=1}^{n} \sigma_k^{*2} \xrightarrow[n\to\infty]{} 1$

(102) $\sum_{k=1}^{n} E|Y_k'|^3 \xrightarrow[n\to\infty]{} 0$

Given.

(a) $$LF_n^{\varepsilon_n} \equiv \sum_{k=1}^{n} \int_{\{|Y| \ge \varepsilon_n\}} y^2 \, dF_k(y) \longrightarrow 0 \quad \text{as } n \to \infty$$

(b) $$\frac{LF_n^{\varepsilon}}{\varepsilon_n^2} \xrightarrow[n\to\infty]{} 0$$

(c) $$\varepsilon_n \xrightarrow[n\to\infty]{} 0$$

Proof (100)

Now $E(Y_k) = 0$

$$\sum_{k=1}^{n} Var(Y_k) = 1$$

$E(Y_k) = 0$

$\therefore \quad E\left\{ Y_k I_{[|Y_k| \geq \varepsilon_n]} + E \; Y_k I_{[|Y_k| < \varepsilon_n]} \right\} = 0$

$\therefore \quad E \cdot Y_k I_{[|Y_k| > \varepsilon_n]} = - E\left(Y_k I_{[|Y_k| \leq \varepsilon_n]} \right)$

$\therefore \quad - \sum_{k=1}^{n} E\left(Y_k I_{\{|Y_k| > \varepsilon_n\}} \right) = \sum_{k=1}^{n} \mu_k^*$

To show $\sum_{k=1}^{n} \mu_k^* \longrightarrow 0$ we show $\left| \sum_{k=1}^{n} \mu_k^* - 0 \right| \longrightarrow 0$

Now to show this we show $\left| \sum_{k=1}^{n} E\left[Y_k I_{\{|Y_k| > \varepsilon_n\}} \right] \right| \longrightarrow 0$

$$\left| \sum_{k=1}^{n} E\left(Y_k I_{\{|Y_k| \geq \varepsilon_n\}} \right) \right| \leq \sum_{k=1}^{n} E \, |Y_k| \, I_{\{|Y_k| > \varepsilon_n\}}$$

$$\leq \sum_{k=1}^{n} E \, |Y_k| \, I_{\{|Y_k| \geq \varepsilon_n\}} \qquad \text{——} (**)$$

Now condit$^{\underline{n}}$ ⓐ $\Rightarrow \sum_{k=1}^{n} E \, |Y_k|^2 I_{\{|Y_k| \geq \varepsilon_n\}} \longrightarrow 0$

ⓑ $\Rightarrow \sum_{k=1}^{n} \dfrac{E \, |Y_k|^2 \, I_{\{|Y_k| \geq \varepsilon_n\}}}{\varepsilon_n^2} \longrightarrow 0 \qquad \text{——} (***)$

Now $|Y_k| \geq \varepsilon_n$

$$\Rightarrow \frac{1}{|Y_k|} \leq \frac{1}{\varepsilon_n} \Rightarrow \frac{|Y_k|^2}{|Y_k|} \leq \frac{|Y_k|^2}{\varepsilon_n}$$

②

$$\therefore \quad |Y_k| \leq \frac{|Y_k|^2}{\varepsilon_n}$$

$$\therefore \quad |Y_k| I_{\{|Y_k| \geq \varepsilon_n\}} \leq \frac{|Y_k|^2}{\varepsilon_n} I_{\{|Y_k| \geq \varepsilon_n\}}$$

$$\sum_{k=1}^{n} E|Y_k| I_{\{|Y_k| \geq \varepsilon_n\}} \leq \sum_{k=1}^{n} \frac{E|Y_k|^2 I_{\{|Y_k| \geq \varepsilon_n\}}}{\varepsilon_n}$$

$$= \varepsilon_n \sum_{k=1}^{n} \frac{E|Y_k|^2 I_{\{|Y_k| \geq \varepsilon_n\}}}{\varepsilon_n^2}$$

$$\longrightarrow 0 \quad \text{as } n \to \infty. \quad \text{by } (***)$$

$\therefore$ $(**) \longrightarrow 0$ also

$$\therefore \quad \left| \sum_{k=1}^{n} \mu_k^* \right| \longrightarrow 0 \quad \text{as } n \to \infty$$

Proof 102 :

$$\sum_{k=1}^{n} E\,|Y_k'|^3 = \sum_{k=1}^{n} E\,\left| Y_k\, I_{\{|Y_k| \le \varepsilon_n\}} \right|^3$$

$$= \sum_{k=1}^{n} E\left(|Y_k|^3\, I_{\{|Y_k| \le \varepsilon_n\}} \right)$$

$$\le \varepsilon_n \sum_{k=1}^{n} E\,|Y_k|^2\, I_{\{|Y_k| \le \varepsilon_n\}}$$

$$\le \varepsilon_n \sum_{k=1}^{n} E\,|Y_k|^2$$

$$= \varepsilon_n \sum_{k=1}^{n} \mathrm{Var}(Y_k) \qquad \{ \because E(Y_k) = 0$$

$$= \varepsilon_n \qquad \{ \because \sum_{k=1}^{n} \mathrm{Var}(Y_k) = 1$$

$$\longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

Proof 101

$$\sum_{k=1}^{n} \sigma_k^{*2} = \sum_{k=1}^{n} Var(Y_k') = \sum_{k=1}^{n} Var\left(Y_k I_{\{|Y_k| \le \varepsilon_n\}} \right)$$

$$= \sum_{k=1}^{n} E\left[(Y_k^2)\, I_{\{|Y_k| \le \varepsilon_n\}}\right] - \sum_{k=1}^{n} \left[E(Y_k)\, I_{\{|Y_k| \le \varepsilon_n\}}\right]^2$$

$$= \sum_{k=1}^{n} E\left[Y_k^2\left(1 - I_{\{|Y_k| > \varepsilon_n\}}\right)\right] - \sum_{k=1}^{n} \left[E\left(Y_k I_{\{|Y_k| \le \varepsilon_n\}}\right)\right]^2$$

$$= \sum_{k=1}^{n} E(Y_k^2) - \sum_{k=1}^{n} E\left(Y_k^2 I_{\{|Y_k| > \varepsilon_n\}}\right) - \sum_{k=1}^{n} \left[E\left(Y_k I_{\{|Y_k| \le \varepsilon_n\}}\right)\right]^2$$

$$= \sum_{k=1}^{n} Var(Y_k) - LF_n^{\varepsilon_n} - \sum_{k=1}^{n} \left[E\left(Y_k I_{\{|Y_k| \le \varepsilon_n\}}\right)\right]^2$$

$$= 1 - LF_n^{\varepsilon_n} - \sum_{k=1}^{n} \left[E\left(Y_k I_{\{|Y_k| \le \varepsilon_n\}}\right)\right]^2$$

Now $E\left(Y_k I_{\{|Y_k| \le \varepsilon_n\}}\right)^2 \le E(Y_k^2)\, E\left(I_{\{|Y_k| \le \varepsilon_n\}}\right)$ $\left[[E(XY)]^2 \le E(X^2)E(Y^2) \right.$

$$= E(Y_k^2)\, P(|Y_k| \le \varepsilon_n)$$

$\{ Var(Y_k) = E(Y_k^2) \le 1$

$$\therefore \sum_{k=1}^{n} E\left(Y_k I_{[|Y_k| \le \varepsilon_n]}\right)^2 \le \sum_{k=1}^{n} E(Y_k^2)\, P(|Y_k| \le \varepsilon_n) \le \sum_{k=1}^{n} P(|Y_k| \le \varepsilon_n)$$

Now $E(Y_k) = 0$

$$\therefore \quad E\left(Y_k I_{\{|Y_k| \le \varepsilon_n\}}\right) + E\left(Y_k I_{\{|Y_k| > \varepsilon_n\}}\right) = 0$$

$$\therefore \quad \left[E\left(Y_k I_{\{|Y_k| \le \varepsilon_n\}}\right)\right]^2 = \left[E\left(Y_k I_{\{|Y_k| > \varepsilon_n\}}\right)\right]^2$$

$$\therefore \sum_{k=1}^{n}\left(E\left(Y_k I_{\{|Y_k| \le \varepsilon_n\}}\right)\right)^2 = \sum_{k=1}^{n} E\left[Y_k I_{\{|Y_k| > \varepsilon_n\}}\right]^2$$

$$\le \sum_{k=1}^{n} E(Y_k)^2 \, P(|Y_k| > \varepsilon_n)$$

$$\le \sum_{k=1}^{n} P(|Y_k| > \varepsilon_n) \qquad \text{——} (****)$$

Now $\dfrac{LF_n^{\varepsilon_n}}{\varepsilon_n^2} \longrightarrow 0$ &

$$\frac{\sum_{k=1}^{n} E|Y_k|^2 I_{[|Y_k| \ge \varepsilon_n]}}{\varepsilon_n^2} \ge \frac{\varepsilon_n^2 \sum_{k=1}^{n} E\left(I_{\{|Y_k| > \varepsilon_n\}}\right)}{\varepsilon_n^2}$$

$$= \frac{\varepsilon_n^2 \sum_{k=1}^{n} P(|Y_k| > \varepsilon_n)}{\varepsilon_n^2}$$

$$\therefore \quad \frac{\cancel{\varepsilon_n^2} \sum_{k=1}^{n} P(|Y_k| > \varepsilon_n)}{\cancel{\varepsilon_n^2}} \le \frac{LF_n^{\varepsilon_n}}{\varepsilon_n^2} \longrightarrow 0$$

$\therefore \quad (****) \longrightarrow 0$

QED

Mills' Ratio [Stuart & Kendell]

$$\left(\frac{1}{x} - \frac{1}{x^3}\right)\phi(x) \leq 1 - \Phi(x) \leq \frac{1}{x}\phi(x) \qquad ; \; x > 0$$

where $$\Phi(x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx$$

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$R(x) = \frac{\Phi(x)}{1-\phi(x)}$$ is called Mills Ratio.

$$\frac{x}{1+x^2} \leq R(x) \leq \frac{1}{x}$$

$$\frac{1}{x} - \frac{1}{x^3} \leq R(x) \leq \frac{1}{x}$$

$$\frac{2}{x + \sqrt{x^2+4}} \leq R(x) \leq \frac{2}{x + \sqrt{x^2+2}}$$

are the commonly found bounds

Proof: $$\int_x^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt \leq \int_x^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \frac{t}{x} \, dt$$

$$= \frac{1}{x} \int_x^{\infty} t \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$$

Let $u = t^2/2 \Rightarrow du = t\,dt$

Range $\frac{x^2}{2}$ to ∞

$$\int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt \leq \frac{1}{x} \int_{x^2/2} \frac{1}{\sqrt{2\pi}} e^{-u} du$$

$$= \frac{1}{x\sqrt{2\pi}} e^{-x^2/2}$$

$$= \frac{1}{x} \phi(x)$$

To show $\left(\frac{1}{x} - \frac{1}{x^3}\right) \phi(x) \leq 1 - \Phi(x)$

Now $\frac{1}{1+u^2} e^{-\frac{x^2(1+u^2)}{2}} = \int_{x^2/2}^\infty e^{-v(1+u^2)} . dv$

$$\therefore \int_{-\infty}^\infty \frac{1}{(1+u^2)} e^{-(x^2/2)(1+u^2)} . du = \int_{-\infty}^\infty \int_{x^2/2}^\infty e^{-v(1+u^2)} . dv\, du$$

$$= \int_{x^2/2}^\infty \int_{-\infty}^\infty e^{-v(1+u^2)} du\, dv$$

$$= \sqrt{2\pi} \int_{x^2/2}^\infty \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-v} . e^{-vu^2} . du\, dv$$

$$= \sqrt{2\pi} \int_{x^2/2}^{\infty} e^{-v} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{u^2}{2\left(\frac{1}{2v}\right)}\right\} du \, dv$$

$$= \sqrt{2\pi} \int_{x^2/2}^{\infty} e^{-v} \sqrt{\frac{1}{2v}} \cdot dv$$

$$= \sqrt{2\pi} \int_{x^2/2}^{\infty} e^{-v} (2v)^{-1/2} \, dv$$

Take $u = \sqrt{2v}$ ie $\frac{u^2}{2} = v$

$\Rightarrow$ $dv = u\,du \Rightarrow dv/\sqrt{2v} = du$

Range x to ∞

$\therefore$

$$= \sqrt{2\pi} \int_{x}^{\infty} e^{-u^2/2} \, du$$

$$= 2\pi \int_{x}^{\infty} \frac{e^{-u^2/2}}{\sqrt{2\pi}} \, du$$

$$= 2\pi \left[1 - \Phi(x)\right]$$

$$\therefore \int_{-\infty}^{\infty} \frac{1}{(1+u^2)} \exp\left\{-\frac{x^2(1+u^2)}{2}\right\} du = 2\pi \left[(1-\Phi(x))\right]$$

We will show $\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{(1+u^2)} \exp\left\{\frac{-1}{2} x^2(1+u^2)\right\} du \geq \left(\frac{1}{x} - \frac{1}{x^3}\right)\phi(x)$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{(1+u^2)} \exp\left\{-\frac{x^2(1+u^2)}{2}\right\} du = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{(1+u^2)} \exp\left\{-\frac{x^2u^2}{2}\right\} du$$

$$= \phi(x) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{(1+u^2)} e^{-\frac{x^2u^2}{2}} du$$

$\therefore$ We need to show $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{(1+u^2)} e^{-x^2u^2/2} du \geq \left(\frac{1}{x} - \frac{1}{x^3}\right)$

Let $t = \frac{x^2u^2}{2} \Rightarrow u = \frac{\sqrt{2t}}{x} \Rightarrow du = \frac{1}{x}\sqrt{2}\,\frac{1}{2} t^{-1/2} dt$

$$du = \frac{1}{\sqrt{2}} \frac{1}{x} t^{-1/2} dt$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{(1+u^2)} e^{-\frac{x^2u^2}{2}} \cdot du = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} \frac{1}{(1+u^2)} e^{-\frac{x^2u^2}{2}} du$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} \exp\{-t\} \frac{1}{\left(1+\frac{2t}{x^2}\right)} \frac{1}{\sqrt{2}} \frac{1}{x} t^{-1/2} dt$$

$$= \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-t} \frac{1}{\frac{2}{x^2}\left(\frac{x^2}{2}+t\right)} \frac{1}{x} \frac{1}{\sqrt{t}} \, dt$$

$$= \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{1}{2} e^{-t} \frac{1}{\frac{\sqrt{t}}{x}\left(x^2/2+t\right)} \, dt$$

$$= \frac{x}{\sqrt{\pi}} \int_0^\infty \frac{1}{\cancel{2}} e^{-t} t^{-1/2} \frac{1}{\frac{x^2}{\cancel{2}}\left(1+\frac{2t}{x^2}\right)} \, dt$$

$$= \frac{x}{\sqrt{\pi}} \int_0^\infty e^{-t} t^{-1/2} \frac{1}{\left(1+\frac{2t}{x^2}\right)} \, dt$$

$$= \frac{x}{\sqrt{\pi}} \int_0^\infty e^{-t} t^{-1/2} \sum_{j=0}^\infty (-1)^j \left(\frac{2t}{x^2}\right)^j \cdot dt$$

$$= \frac{x}{\sqrt{\pi}} \sum_{j=0}^\infty \frac{(-1)^j}{x^{2j+2}} \int_0^\infty e^{-t} 2^j t^{j+1/2-1} \cdot dt$$

$$= \frac{x}{\sqrt{\pi}} \sum_{k=0}^\infty \frac{(-1)^k}{x^{2k+2}} \int_0^\infty e^{-t} 2^k t^{k+1/2-1} \cdot dt$$

$$= \frac{x}{\sqrt{\pi}} \left\{ \underbrace{\sum_{k=0}^{j} \frac{1}{x^{2k+2}} (-1)^k \int_0^\infty e^{-t} 2^k t^{k+\frac{1}{2}-1} \, dt}_{A} + \underbrace{\sum_{k=j+1}^{\infty} \frac{(-1)^k}{x^{2k+2}} \int_0^\infty e^{-t} 2^k t^{k+\frac{1}{2}-1} \, dt}_{B} \right\}$$

$$= \sum_{k=0}^{j} \frac{(-1)^k}{x^{2k+1}} \left(\frac{1}{\sqrt{\pi}}\right) \int_0^\infty e^{-t} 2^k t^{k+\frac{1}{2}-1} \cdot dt \quad + \quad B$$

Now

$$A = \frac{1}{x}\frac{1}{\sqrt{\pi}} \int_0^\infty e^{-t} t^{\frac{1}{2}-1} \cdot dt \; - \frac{1}{x^3}\frac{1}{\sqrt{\pi}} \int_0^\infty e^{-t}\, 2\, t^{3/2-1} dt \; + \frac{1}{x^5}\frac{1}{\sqrt{\pi}} \int_0^\infty e^{-t}\, 4\, t^{5/2-1} dt \ldots$$

$$= \frac{1}{x}\frac{1}{\sqrt{\pi}} \Gamma(1/2) \; - \frac{1}{x^3}\frac{2}{\sqrt{\pi}} \Gamma(3/2) + \frac{1}{x^5}\frac{4}{\sqrt{\pi}} \Gamma(5/2) \ldots$$

$$= \frac{1}{x}\frac{1}{\sqrt{\pi}}\sqrt{\pi} \; - \frac{1}{x^3}\frac{2}{\sqrt{\pi}}\frac{1}{2}\sqrt{\pi} + \frac{1}{x^5}\frac{\cancel{4}}{\cancel{\sqrt{\pi}}}\frac{3}{\cancel{2}}\frac{1}{\cancel{2}}\cancel{\sqrt{\pi}} \ldots$$

$$= \left(\frac{1}{x} - \frac{1}{x^3} + \frac{3}{x^5} - \ldots \right)$$

The j^{th} term of A is $\quad \dfrac{(-1)^j}{x^{2j+1}} \dfrac{1}{\sqrt{\pi}} \displaystyle\int_0^\infty e^{-t} 2^j t^{j+1/2-1} \cdot dt$

$$= \frac{(-1)^j (2)^j}{x^{2j+1}} \frac{1}{\sqrt{\pi}} \Gamma(j+1/2)$$

$$= \frac{(-2)^j}{x^{2j+1}} \frac{1}{\sqrt{\pi}} \left(j+\tfrac{1}{2}-1\right)\left(j+\tfrac{1}{2}-2\right) \cdots \frac{3}{2}\;\frac{1}{2}\;\sqrt{\pi}$$

$$= \frac{(-1)^j (2)^j}{x^{2j+1}} \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}+1\right) \cdots \left(\tfrac{1}{2}+j-2\right)\left(\tfrac{1}{2}+j-1\right)$$

$$= \frac{(-1)^j (2)^j}{x^{2j+1}} \left(\frac{(1)(3)\cdots(2j-3)(2j-1)}{2^j}\right)$$

$$= \frac{(-1)^j}{x^{2j+1}} \quad 1\cdot 3\cdot 5\cdot \cdots \cdot (2j-1)$$

$$|B| = \left|\left(\frac{x}{\sqrt{\pi}}\right) \sum_{k=j+1}^{\infty} \left[\frac{(-1)^k}{x^{2k+2}}\right] \int_0^{\infty} e^{-t}\, 2^k\, t^{k+1/2-1}\, dt\right|$$

$$= \left|\frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-t}\, \frac{t^{-1/2}}{x} \sum_{k=j+1}^{\infty} \left(\frac{2t}{x^2}\right)^k \cdot dt\right|$$

$$= \left|\frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-t}\, \frac{t^{-1/2}}{x} \left(\frac{2t}{x^2}\right)^{j+1} \left[1 + \left(\frac{2t}{x^2}\right)^1 + \left(\frac{2t}{x^2}\right)^2 + \cdots\right] \cdot dt\right|$$

$$= \left| \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-t} \frac{t^{-1/2}}{x} \frac{(2t)^{j+1}}{x^{2j+2}} (-1)^{j+1} \frac{1}{\left[1+\left(\frac{2t}{x^2}\right)\right]} dt \right|$$

$$= \left| \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-t} \frac{t^{-1/2}}{x} \frac{(2t)^{j+1}(-1)^{j+1}}{x^{2j+2}} \frac{1}{\frac{2}{x^2}\left(\frac{x^2}{2}+t\right)} dt \right|$$

$$= \left| \frac{1}{2\sqrt{\pi}} \int_0^\infty e^{-t} t^{-1/2} \frac{(2t)^{j+1}(-1)^{j+1}}{x^{2j+1}} \cdot \frac{1}{\left(\frac{x^2}{2}+t\right)} \cdot dt \right|$$

$$\leq \left| 2\frac{1}{2\sqrt{\pi}} \int_0^\infty e^{-t} 2^{j+1} \frac{1}{x^{2j+1}} t^{j+1/2-1} \cdot dt \right| \qquad \left\{ \because \frac{1}{\left(\frac{x^2}{2}+t\right)} \leq \frac{1}{t} \right.$$

$$= \left| 2\frac{1}{2\sqrt{\pi}} \int_0^\infty e^{-t} 2^{j+1} \frac{1}{x^{2j+1}} t^{j+1/2-1} dt \right|$$

$$= \left| \frac{1}{x^{2j+1}} \left(\frac{2^{j+1}}{2\sqrt{\pi}} \right) \int_0^\infty e^{-t} t^{j+1/2-1} dt \right|$$

$$= \left| \frac{2^{j+1}}{2\sqrt{\pi}} \left(\frac{1}{x^{2j+1}} \right) \Gamma\left(j+\tfrac{1}{2}\right) \right|$$

$$= \frac{1}{x^{2j+1}} \frac{2^{j+1}}{2\cancel{\sqrt{\pi}}} (j+\tfrac{1}{2}-1)(j+\tfrac{1}{2}-2) \cdots (\tfrac{3}{2})(\tfrac{1}{2})(\cancel{\sqrt{\pi}})$$

$$= \frac{1}{x^{2j+1}} \; 2^{j} \; \frac{(2j-1)(2j-3)\cdots(3)(1)}{2^{j}}$$

$|B| \leq j^{th}$ term of A

$\therefore \left(\frac{1}{x} - \frac{1}{x^3}\right) \geq A$

QED