1) The r.v $X_1, X_2 \cdots X_k$ are indep $\iff$ $\Phi_{\underset{\sim}{X}}(t_1, \cdots t_k) = \prod_{j=1}^{k} \Phi_{X_j}(t_j)$

Proof: Given $X_1, X_2 \cdots X_k$ are indep

$\Rightarrow$ $e^{it_jX_j}$'s are also indep $\forall j = 1, 2 \cdots k$

$$\Phi_{\underset{\sim}{X}}(\underset{\sim}{t}) = E\left(e^{it^TX}\right)$$

$$= E\left[e^{i(t_1X_1 + \cdots + t_kX_k)}\right] \qquad \{\text{definition}$$

$$= E\left(e^{it_1X_1} e^{it_2X_2} \cdots e^{it_kX_k}\right)$$

$$= E(e^{it_1X_1}) E(e^{it_2X_2}) \cdots E(e^{it_kX_k}) \qquad \{\text{by indep}$$

$$= \Phi_{X_1}(t_1) \Phi_{X_2}(t_2) \cdots \Phi_{X_k}(t_k)$$

$$= \prod_{j=1}^{k} \Phi_{X_j}(t_j) \qquad \text{———— } (*)$$

Now given $\Phi_{\underset{\sim}{X}}(t) = \prod_{j=1}^{k} \Phi_{X_j}(t_j)$

Let $Y_1, Y_2 \cdots Y_k$ be indep r.v. st $Y_j \simeq X_j$ (can always do this.

$$\Phi_{\underset{\sim}{Y}}(\underset{\sim}{t}) = E\left(e^{i\underset{\sim}{t}^T\underset{\sim}{Y}}\right) = \prod_{j=1}^{k} \Phi_{Y_j}(t_j) \qquad \text{from } (*)$$

$$= \prod_{j=1}^{k} \Phi_{X_j}(t_j) \qquad \because Y_j \simeq X_j$$

$$= \Phi(t) \qquad \{\text{given}$$

$\therefore \quad \underset{\sim}{y} \simeq \underset{\sim}{X}$

$\Rightarrow \quad X_1, X_2 \cdots X_k$ are also indep

Ex 2] (a) Derive the poisson chf

$$\Phi_X(t) = E(e^{itX}) = \sum_{x=0}^{\infty} e^{itx} \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(e^{it} \lambda)^x}{x!}$$

$$= e^{-\lambda} e^{\lambda e^{it}}$$

$$= e^{\lambda (e^{it} - 1)}$$

(b) Derive the chf of the geom(p)

$$\Phi_X(t) = E(e^{itX}) = \sum_{x=0}^{\infty} e^{itx} q^x p$$

$$= p \sum_{x=0}^{\infty} (q e^{it})^x$$

$$= \frac{p}{(1 - q e^{it})}$$

(c) Bernoulli$(p)$

$$\Phi_X(t) = E(e^{itX}) = pe^{it} + qe^{0} = q + pe^{it}$$

(d) Bin$(n,p)$

$$\Phi_X(t) = \sum_{x=0}^{n} e^{itx} \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (pe^{it})^x (1-p)^{n-x}$$

$$= (q + pe^{it})^n$$

(e) NegBin$(m,p)$

$$\Phi_X(t) = \sum_{x=0}^{\infty} e^{itx} \binom{-m}{x} p^m (-q)^x$$

$$= p^r \sum_{x=0}^{\infty} \binom{-m}{x} (-qe^{it})^x$$

$$= p^r (1-qe^{it})^{-m}$$

$$= \left(\frac{p}{1-qe^{it}}\right)^m$$

3] (a) Derive Gamma$(r, \theta)$ chf.

$$\Phi_X(t) = E(e^{itx}) = \int_0^\infty e^{itx} \, e^{-\theta x} \, x^{r-1} \, \frac{\theta^r}{\Gamma(r)} \, dx$$

$$= \frac{\theta^r}{\Gamma(r)} \int_0^\infty e^{-(\theta - it)x} \, x^{r-1} \, dx$$

$$= \frac{\theta^r}{\Gamma(r)} \, \frac{\Gamma(r)}{(\theta - it)^r}$$

$$= \frac{\theta^r}{\theta^r (1 - it/\theta)^r}$$

$$= \left(1 - \frac{it}{\theta}\right)^{-r}.$$

4]
$$\left.\begin{aligned} \sqrt{m}\,(S_m - \theta) &\xrightarrow[m\to\infty]{d} N(0,1) \\ \sqrt{n}\,(T_n - \theta) &\xrightarrow[n\to\infty]{d} N(0,1). \end{aligned}\right\} \Rightarrow \sqrt{\frac{mn}{m+n}}\,(S_m - T_n) \xrightarrow[\substack{n\to\infty \\ m\to\infty}]{d} N(0,1)$$

where $S_m$ and $T_n$ are indep and we assume that

$$\left(\frac{m}{m+n}\right) \xrightarrow[n\to\infty]{} \lambda \qquad \text{for some } \lambda \in [0,1]$$

4] Given

$$\left.\begin{array}{l}\sqrt{m}\,(S_m-\theta) \xrightarrow[m\to\infty]{d} N(0,1) \\ \sqrt{n}\,(T_n-\theta) \xrightarrow[n\to\infty]{d} N(0,1)\end{array}\right\} \Rightarrow \sqrt{\frac{mn}{m+n}}\,(S_m-T_m) \xrightarrow[\substack{n\to\infty \\ m\to\infty}]{d} N(0,1)$$

$$\sqrt{\frac{mn}{m+n}}\,(S_m-T_n) = \sqrt{\frac{mn}{m+n}}\,(S_m-\theta-T_n+\theta)$$

$$= \underbrace{\sqrt{m}\,(S_m-\theta)}_{\xrightarrow{d} N(0,1)}\underbrace{\sqrt{\frac{n}{n+m}}}_{\to \sqrt{1-\lambda}} - \underbrace{\sqrt{n}\,(T_n-\theta)}_{\xrightarrow{d} N(0,1)}\underbrace{\sqrt{\frac{m}{m+n}}}_{\to \sqrt{\lambda}}$$

$$\xrightarrow{d} \sqrt{1-\lambda}\,(Z) - \sqrt{\lambda}\,Z$$

$\left\{\begin{array}{l} Z \sim SN(0,1) \\ \text{by slutsky}\end{array}\right.$

Let $V = \sqrt{1-\lambda}\,Z - \sqrt{\lambda}\,Z$

$V \sim N(0,1)$

- qed

Shuva's Recitation - WED.

$\equiv \{ X \text{ on } (\Omega, \mathcal{A}, P) : E(X^2) < \infty \quad E(X) = 0 \}$

$\mathcal{F}(X_1) \equiv \{ X_M \text{ on } (\Omega, \mathcal{A}, P) \text{ which are } \mathcal{F}(X_1) \text{ meas. with } E(X)=0, E(X^2)<\infty \}$

$= \{ g(X_1) \text{ which are } \mathcal{F}(X_1)\text{-meas} \}$

$\mathcal{D} \subset \mathcal{A}$ is a sub $\sigma$-field

$\mathcal{H}_{\mathcal{D}} = \{ X \text{ on } (\Omega, \mathcal{A}, P) \text{ which are } \mathcal{D}\text{-meas st } E(X)=0 \quad E(X^2) < \infty \}$

$\mathcal{H}$ is a vector space

$P_{\mathcal{D}}(y) \equiv E(Y | \mathcal{D}) \quad \forall\, Y \in \mathcal{H}$ is a linear funct$^{n}$ on $\mathcal{H}$

$\langle x, y \rangle \equiv E(XY)$ is a inner prod defined on $\mathcal{H}$

$\mathcal{H}_{\mathcal{D}}$ and $\mathcal{H}_{\mathcal{F}(X_1)}$ are subspaces of $\mathcal{H}$

<u>Note</u> If we did not have $E(X)=0$ in def$^{n}$ of $\mathcal{H}$ then we need to define $\langle x, y \rangle = E[(X-E(X))(Y-E(Y))]$

We show $P_{\mathscr{D}}(\circ)$ is a ppo

(i) $P_{\mathscr{D}}^2 = P_{\mathscr{D}}$ $\left\{ \begin{array}{l} P_{\mathscr{D}}(P_{\mathscr{D}}(Y)) = E(E(Y|\mathscr{D})\,|\,\mathscr{D}) \underset{\uparrow}{=} E(Y|\mathscr{D}) = P_{\mathscr{D}} \\ \text{If } X \text{ is } \mathscr{D}\text{-meas then } E(X|\mathscr{D}) = X \end{array} \right\}$

(ii). $(I - P_{\mathscr{D}})$ is a proj on $\mathcal{H}_{\mathscr{D}}^{\perp}$

$$\langle (I - P_{\mathscr{D}})Y \,,\, P_{\mathscr{D}} Y \rangle = 0$$

(iii).

$$\langle P_{\mathscr{D}} Y , Z \rangle = \langle Y , P_{\mathscr{D}} Z \rangle$$

ACE (Lec 34 - Pg 6)

$\mathcal{H}_1 \equiv \mathcal{H}_{\mathcal{F}(X_1)} = \{ \mathcal{F}(X_1) \text{ meas funct}^n \text{ st } E(X_1) = 0 \text{ \& } E(X_1^2) < \infty \}$

$\mathcal{H}_2 \equiv \mathcal{H}_{\mathcal{F}(X_2)}$

$P_1(Y) \equiv E(Y|X_1)$ (is funct$^n$ from $\mathcal{H}$ to $\mathcal{H}_1$)

$P_2(Y) \equiv E(Y|X_2)$ (is funct$^n$ from $\mathcal{H}$ to $\mathcal{H}_2$)

$\mathcal{H}_+ \equiv \mathcal{H}_1 + \mathcal{H}_2$

$\equiv \{ g_1(X_1) + g_2(X_2) : g_i \text{ is meas and } E(g_i^2(X_i)) < \infty \}$

}

We say $\mathcal{H}_1 \perp \mathcal{H}_2$ if $\langle g_1(X_1) ; g_2(X_2) \rangle \equiv E(g_1(X_1) g_2(X_2)) = 0$

for any $g_1, g_2$

Property 1

$X_1$ and $X_2$ are indep $\Leftrightarrow$ $\mathcal{H}_1 \perp \mathcal{H}_2$

Proof : $X_1$ and $X_2$ are indep

$$\Rightarrow E[g_1(X_1) g_2(X_2)] = E[g_1(X_1)] E[g_2(X_2)]$$

Consider

$$\begin{aligned}
\langle g_1(X_1) ; g_2(X_2) \rangle &= E\left[\left(g_1(X_1) - E(g_1(X_1))\right)\left(g_2(X_2) - E(g_2(X_2))\right)\right] \\
&= E[g_1(X_1) g_2(X_2)] - E(g_1(X_1)) E(g_2(X_2)) \\
&= E[g_1(X_1)] E[g_2(X_2)] - E[g_1(X_1)] E[g_2(X_2)] \\
&= 0
\end{aligned}$$

Pg ③

Property: $\langle (I-P_{\mathcal{D}})Y \,;\, P_{\mathcal{D}} Y \rangle = 0$.

$\langle (I-P_{\mathcal{D}})Y \,;\, h \rangle = 0$ for any $h$ which is $\mathcal{D}$-meas

Proof

$$\langle (I-P_{\mathcal{D}})Y \,;\, P_{\mathcal{D}} Y \rangle = \langle Y - E(Y|\mathcal{D}) ; E(Y|\mathcal{D}) \rangle$$

$$= E\left[\left(Y - E(Y|\mathcal{D})\right) E(Y|\mathcal{D})\right]$$

$$= E\left[ Y E(Y|\mathcal{D}) - E(Y|\mathcal{D}) E(Y|\mathcal{D})\right]$$

$$= E\left[ Y E(Y|\mathcal{D})\right] - E^2(Y|\mathcal{D})$$

$$= \ ?$$

NOTE

$\langle Y - E(Y|X_1) , X_1 \rangle = 0$ for any $Y$ and $X_1$

$\boxed{\text{To show}}$ $X_1$ indep of $X_2$ $\iff$ $\mathcal{H}_1 \perp \mathcal{H}_2$

Proof: $\langle g_2(X_2) - E(g_2(X_2)/g_1(X_1)) \,;\, g_1(X_1) \rangle = 0$ always true

$\Rightarrow$ $\langle g_2(X_2); g_1(X_1) \rangle - \langle E[g_2(X_2) | g_1(X_1)] \,;\, g_1(X_1) \rangle = 0$

Given $X_1$ indep of $X_2$

$$\Rightarrow \langle g_2(X_2), g_1(X_1) \rangle = \langle E[g_2(X_2)] \,;\, g_1(X_1) \rangle$$

$$= E\left\{\left([g_1(X_1) - E(g_1(X_1))]\left[E(g_2(X_2)) - E[g_2(X_2)]\right]\right)\right\}$$

$$= E\left[(g_1(X_1) - E(g_1(X_1)))(0)\right]$$

$$= 0$$

$\rightarrow$ $\mathcal{H}_1 \perp \mathcal{H}_2$

Now given $\mathcal{H}_1 \perp \mathcal{H}_2$ $\Rightarrow$ $\langle g_1(X_1); g_2(X_2) \rangle = 0$

$$\Rightarrow E\left[g_1(X_1) - E(g_1(X_1))\right]\left[g_2(X_2) - E(g_2(X_2))\right] = 0$$

$$\Rightarrow E[g_1(X_1) g_2(X_2)] - E[g_1(X_1)]E[g_2(X_2)] = 0$$

$$\Rightarrow E[g_1(X_1)\, g_2(X_2)] = E[g_1(X_1)]\, E[g_2(X_2)]$$

$\Longrightarrow$ $X_1$ and $X_2$ are indep

<u>Property</u> $E\left[Y \mid g(X_1), g(X_2)\right] = E\left[Y \mid g(X_1)\right] + E\left[Y \mid g(X_2)\right]$

if $X_1$ and $X_2$ are indep

To show $\underset{\sim}{v} = 0$ show $\langle v, v \rangle = 0$

$\langle v, v \rangle = 0 \iff \langle v, w \rangle = 0 \quad \forall \underset{\sim}{w}$

<u>Proof</u>

$\mathcal{H}_1 \equiv \mathcal{H}_{\mathcal{F}(X_1)}$

$\mathcal{H}_2 \equiv \mathcal{H}_{\mathcal{F}(X_2)}$

$P_1(\cdot) \equiv E(\cdot \mid X_1)$

$P_2(\cdot) \equiv E(\cdot \mid X_2)$

$\mathcal{H}_+ \equiv \mathcal{H}_1 + \mathcal{H}_2 = \{g_1(X_1) + g_2(X_2)\}$

$P_+$ is a projection onto $\mathcal{H}_+$ $\{$ie $\langle Y - P_+ Y, P_+ Y \rangle = 0$ $\}$

To show $P_+ = P_1 + P_2$

ie show $P_+(Y) = P_1(Y) + P_2(Y)$

show $P_+(Y) = E(Y \mid X_1) + E(Y \mid X_2)$

~~We need to show~~ $\cancel{\langle (I - P_+)(Y) ; Z \rangle = 0 \quad \forall Z \in \mathcal{H}_+}$

We need to show $\langle (I - P_1 - P_2) Y ; Z \rangle = 0 \quad \forall Z \in \mathcal{H}_+$

Consider

$$\langle \underline{Y - E(Y|X_1)} - \underline{E(Y|X_2)}, Z \rangle = \langle Y - E(Y|X_1), Z \rangle - \langle E(Y|X_2), Z \rangle$$

$$= \langle Y, Z \rangle - \langle E(Y|X_1), Z \rangle - \langle E(Y|X_2), Z \rangle$$

SHUVA'S FRIDAY RECITATION



$$\frac{1}{\theta^r \Gamma(r)} \int_0^\infty \lim_{k\to\infty} \sum_{n=0}^{k} \frac{(it)^n}{n!} x^n \, e^{-x/\theta} \, x^{r-1} \, dx$$

$$= \lim_{k\to\infty} \sum_{n=0}^{k} \frac{1}{\theta^r \Gamma(r)} \int_0^\infty \frac{(it)^n}{n!} x^n \, e^{-x/\theta} \cdot x^{r-1} \cdot dx$$

2]

$$\int_{\sum_{j\in I_0} D_j} \sum_i \frac{P(AD_i)}{P(D_i)} I_{D_i}\, dP = \sum_{i\in I}\left( \int_{\sum_{j\in I_0} D_j} \frac{P(AD_i)}{P(D_i)} I_{D_i}\, dP \right) \quad \left\{ \text{MCT} \right.$$

$$= \sum_{i\in I_0} (\quad) + \sum_{i\notin I_0} (\quad)$$

$$\sum_{i\notin I_0} (\quad) = \sum_{i\notin I_0} \int_{\Omega} \frac{P(AD_i)}{P(D_i)} I_{\left(D_i \cap \sum_{j\in I_0} D_j\right)}\, dP$$

$$= \sum_{i\notin I_0} \int_{\Omega} \frac{P(AD_i)}{P(D_i)} I_{(\phi)}\, dP = 0$$

$$\sum_{i\in I_0} (\quad) = \sum_{i\in I_0} \int_{\sum_{j\in I_0} D_j} \frac{P(AD_i)}{P(D_i)} I_{D_i}\, dP$$

$$= \sum_{i\in I_0} \int_{\Omega} \frac{P(AD_i)}{P(D_i)} I_{D_i \cap \sum_{j\in I_0} D_j}\, dP$$

$$= \sum_{i\in I_0} \int_{\Omega} \frac{P(AD_i)}{P(D_i)} I_{D_i}\, dP \quad \left\{ \begin{array}{l} D_i \cap \sum_{j\in I_0} D_j = D_i \\ \text{as } i \in I_0 \\ \text{\& } D_j\text{'s are disjoint} \end{array} \right.$$

$$\therefore \int_{\sum_{j \in I_0} D_j} \sum_{i \in I} \frac{P(AD_i)}{P(D_i)} I_{D_i} \, dP = \sum_{i \in I_0}{}' \int_{\Omega} \frac{P(AD_i)}{P(D_i)} I_{D_i} \, dP$$

$$= \sum_{i \in I_0}{}' \frac{P(AD_i)}{P(D_i)} \int_{\Omega} I_{D_i} \, dP$$

$$= \sum_{i \in I_0} P(AD_i)$$

$$= \sum_{i \in I_0} \int_{D_i} E(I_A \mid \mathscr{D}) \, dP$$

$$= \sum_{i \in I_0} \int I_{D_i} E(I_A \mid \mathscr{D}) \, dP$$

MCT

$$= \int \sum_{i \in I_0} (I_{D_i}) \, E(I_A \mid \mathscr{D}) \, dP$$

$$= \int I_{\sum_{i \in I_0} D_i} \, E(I_A \mid \mathscr{D}) \, dP$$

$$= \int_{\sum_{i \in I_0} D_i} E(I_A \mid \mathscr{D}) \, dP$$

$$= \int_{\sum_{i \in I_0} D_i} P(A \mid \mathscr{D}) \, dP \qquad \left\{ \text{by def}^n \text{ of conditional prob} \right.$$

4.1] (b) To show $E(Y|\mathcal{D}) = \sum_i \left\{ \frac{1}{P(D_i)} \int_{D_i} Y\,dP \right\} I_{D_i}$

if $P(D_i) = 0$ we define $\left\{ \frac{1}{P(D_i)} \int_{D_i} Y\,dP \right\} = 0$

Proof: We need to show

$$\int_D E(Y|\mathcal{D})\,dP = \int_D \sum_{i\in I} \left\{ \frac{1}{P(D_i)} \int_{D_i} Y\,dP \right\} I_{D_i}\,dP$$

We still have $\Omega = \sum D_i$ & $\mathcal{D} = \sigma[D_1, \dots D_k \dots]$

Case I $Y \geq 0$

$$\text{RHS} = \int_{\sum_{j\in I_0} D_j} \sum_{i\in I} \left\{ \frac{1}{P(D_i)} \int_{D_i} Y\,dP \right\} I_{D_i}\,dP$$

$$= \sum_{i\in I} \int_{\sum_{j\in I_0} D_j} \left( \frac{1}{P(D_i)} \int_{D_i} Y\,dP \right) I_{D_i}\,dP$$

Assume $Y \geq 0$ MCT

$$= \sum_{j\in I_0} \int_{D_j} Y\,dP$$

{as in part (a)

$$= \int_\Omega \Big( \sum_{j\in I_0} I_{D_j} \Big) Y\,dP$$

MCT

$$= \int I_{\sum_{j\in I_0} D_j} \cdot Y \, dP$$

$$= \int_{\sum_{j\in I_0} D_j} Y \, dP$$

$$= \int_{\sum_{j\in I_0} D_j} E(Y \mid \mathcal{D}) \, dP$$

Case II

Now for general $Y = Y^+ - Y^-$

$$E(Y^+ \mid \mathcal{D}) \, dP = \sum_i \left( \frac{1}{P(D_i)} \int_{D_i} Y^+ dP \right) I_{D_i}$$

$$E(Y^- \mid \mathcal{D}) \, dP = \sum_i \frac{1}{P(D_i)} \int_{D_i} Y^- dP \; I_{D_i}$$

$$\therefore \; E(Y^+ - Y^- \mid \mathcal{D}) = E(Y^+ \mid \mathcal{D}) - E(Y^- \mid \mathcal{D})$$

$$= \sum_{i\in I} \frac{1}{P(D_i)} \left( \int_{D_i} Y^+ dP - \int_{D_i} Y^- dP \right) I_{D_i} \qquad \left\{ \begin{array}{l} Y \in \mathcal{L}_1 \\ \Rightarrow Y^+ \in \mathcal{L}_1 \\ \quad Y^- \in \mathcal{L}_1 \end{array} \right.$$

$$= \sum_{i\in I} \frac{1}{P(D_i)} \int_{D_i} (Y^+ - Y^-) \, dP \; I_{D_i}$$

4.3] $Y : (\Omega, \mathcal{A}, P)$

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| $Y =$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| with prob = | $1/32$ | $2/32$ | $3/32$ | $4/32$ | $15/32$ | $4/32$ | $1/32$ | $2/32$ |

$\mathcal{C} \equiv \mathcal{F}(Y)$

$C_i \equiv [Y = i]$ $\qquad p_i = P(C_i)$

$\mathcal{D} = \sigma\{ C_1 + C_2 ; C_2 + C_6 , C_3 + C_7 , C_4 + C_8 \} \equiv \sigma\{ D_1, D_2, D_3, D_4 \}$

$\mathcal{E} \equiv \sigma\{ C_1 + C_2 + C_5 + C_6 , C_3 + C_4 + C_7 + C_8 \}$

$\mathcal{F} \equiv$

a) $$E(Y) = \int_\Omega Y \, dP = \int_{\sum C_i} Y \, dP = \sum_{i=1}^{8} \int_{C_i} Y \, dP$$

$$= \sum_{i=1}^{8} \int_{C_i} i \, dP = \sum_{i=1}^{8} i \, P(C_i) = \frac{152}{32}.$$

(b) $E(Y|\mathcal{D})$ is a r.v. st $\int_D E(Y|\mathcal{D}) \, dP = \int_D Z \, dP$

$\mathcal{D}$ is $\sigma$-field generated by disjoint unions of $D_i$'s so any $\mathcal{D}$-meas funct$^n$ takes constant values on these $D_i$'s.

$$\int_{C_1 \cap C_5} E(Y|\mathcal{D}) = \int_{C_1+C_5} a_1 \, dP = a_1 P(C_1 + C_5)$$

$$\Rightarrow \int_{C_1+C_5} Y \, dP = a_1 \left(\frac{16}{32}\right)$$

$$\Rightarrow \frac{1}{32} + \frac{5 \times 15}{32} = a_1 \left(\frac{16}{32}\right)$$

$$\Rightarrow \frac{76}{32} = a_1 \left(\frac{16}{32}\right)$$

$$\Rightarrow a_1 = \frac{76}{16}$$

$$Z(\omega) = \begin{cases} 76/16 & ; \omega \in C_1 + C_2 \\ & ; \omega \in C_2 + C_6 \\ & ; \omega \in C_3 + C_7 \\ & ; \omega \in C_4 + C_8 \end{cases}$$

<u>Note</u> You can also use previous Ex. 4.1

Now $E(E(Y|\mathcal{D})) = \frac{76}{16} P(C_1+C_5) + \frac{28}{6} P(C_2+C_6) + 4 \, P(C_3+C_7) + \frac{32}{6} P(C_4+C_8)$

(c) Same as part (b).

(d) $E(Y|\mathcal{F})$

$E(Y|\mathcal{F})$ is a $\mathcal{F}$ meas functⁿ

$\mathcal{F} = \{\phi, \Omega\}$ so a functⁿ meas wrt $\mathcal{F}$ is constant

$$\int_A E(Y|\mathcal{F})\,dP = \int_A Z\,dP \qquad \text{where } A \in \mathcal{F}$$

$$= a_1 \int_A dP$$

Now $$\int_\phi E(Y|\mathcal{F})\,dP = \int_\phi Z\,dP = 0$$

Also $$\int_\Omega E(Y|\mathcal{F})\,dP = \int_\Omega Z\,dP$$

$$\Rightarrow \int_\Omega Y\,dP = \int_\Omega Z\,dP$$

$$\Rightarrow E(Y) = a_1 P(\Omega)$$

$$= a_1$$

$$\Rightarrow a_1 = E(Y|\mathcal{F}) = E(Y).$$

# 14.2.12]

$$Y_{kn} = \alpha + \beta \cancel{x}_k + \varepsilon_k \qquad ; \ \varepsilon_k \sim iid(0, \sigma^2)$$

(a) Obt LSE for $\alpha, \beta$.

(b) Show
$$\begin{bmatrix} \sqrt{n}(\hat{\alpha} - \alpha) \\ \sqrt{S_{xx}}\,(\hat{\beta} - \beta) \end{bmatrix} \xrightarrow{d} N\left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} \right)$$

We will try Cramer-Wald device

$$\begin{bmatrix} \sqrt{n}(\hat{\alpha} - \alpha) \\ \sqrt{S_{xx}}\,(\hat{\beta} - \beta) \end{bmatrix} = \begin{pmatrix} \sqrt{n} & 0 \\ 0 & \sqrt{S_{xx}} \end{pmatrix} \begin{pmatrix} \hat{\alpha} - \alpha \\ \hat{\beta} - \beta \end{pmatrix}$$

$$\left\{ (X^TX)^{-1}X^TY = \begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} \quad a \equiv \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right.$$

$$= \begin{pmatrix} \sqrt{n} & 0 \\ 0 & \sqrt{S_{xx}} \end{pmatrix} \left( (X^TX)^{-1}X^TY - \underset{\sim}{a} \right)$$

$$\left\{ Y = aX + \varepsilon \right.$$

$$= \begin{pmatrix} \sqrt{n} & 0 \\ 0 & \sqrt{S_{xx}} \end{pmatrix} \left[ \underset{\sim}{a} + (X^TX)^{-1}X^T\varepsilon - \underset{\sim}{a} \right]$$

$$= \begin{pmatrix} \sqrt{n} & 0 \\ 0 & \sqrt{S_{xx}} \end{pmatrix} \left( (X^TX)^{-1}X^T\varepsilon \right)$$

$$= \begin{pmatrix} \sum_{k=1}^{n} \theta_k \varepsilon_k \\ \sum_{k=1}^{n} \gamma_k \varepsilon_k \end{pmatrix}$$

Pg 283 ] Ex 6.4 Weighted sum of iid r.v