Midterm 1 : STA 5447

February 23rd 2006

34/35 + 3

- Please write your name here.

  MAHTAB MARKER

- Please provide <u>detailed</u> answers to all questions.

- Maximum number of points = 35.

I Definitions and examples (8 points)

8/8

(a) Let ν_1 and ν_2 be two measures on a σ-field.

2 1) When is ν_1 absolutely continuous w.r.t ν_2?

2) When is ν_1 singular w.r.t ν_2?

(b) Give an example of two measures satisfying a-1) and

1 of two measures satisfying a-2).

(No proof required).

(c) Define:

1) The product space of two measurable spaces. 1

2) The product measures. 1

(d) Let F be a distribution function. Define its inverse. (1)

(e) Let $\{X_n\}_n$ and $\{Z_n\}_n$ be two sequences of random variables. When do we say that they are Khinchin-equivalent? (1)

(f) Give an example of two sequences that are Khinchin equivalent. (1)
(No proof required).

(I) (a) If ν_1 and ν_2 are 2 meas on a σ-field $\mathcal{A}$ we say

1) ν_1 is abs continuous wrt ν_2 denoted by $\nu_1 << \nu_2$

if $\nu_2(A) = 0 \implies \nu_1(A) = 0$ for any $A \in \mathcal{A}$

2) ν_1 is singular wrt ν_2 denoted by $\nu_1 \perp \nu_2$

if $\exists$ a $N \in \mathcal{A}$ st $\nu_1(N) = 0$ and $\nu_2(N^c) = 0$

(b) If ν_2 is λ the lebesgue meas then

Let $X \sim \text{Normal}(\mu, \sigma^2)$ with induced meas P_X

$Y \sim \exp(\gamma)$ with induced meas P_Y

$\lambda_0(A) \equiv \lambda(A \cap [0,1])$ lebesgue meas restricted to the intv $[0,1]$

$Z \sim \text{Pois}(\theta)$ with induced meas P_Z

$W \sim \text{Bin}(n,p)$ with induced meas P_W

$\mu \equiv$ counting meas on $\mathbb{Z}$ (the set of integers $0, 1, 2 \ldots$)

then

$P_X << \lambda$ $\quad$ $P_Z \perp \lambda$

$P_Y << \lambda$ $\quad$ $P_W \perp \lambda$

$\lambda_0 << \lambda$ $\quad$ $\mu \perp \lambda$

(c) 1). Product Space:

if $(\Omega, \mathcal{A})$ and $(\Omega', \mathcal{A}')$ are 2 measurable spaces

let $\mathcal{F} = \{ \sum_{i=1}^{m} A_i \times A_i' \ : m \geq 1 \ , A_i \in \mathcal{A} \ , A_i' \in \mathcal{A}' \}$

and $\mathcal{A} \times \mathcal{A}' = \sigma[\mathcal{F}]$

then $(\Omega \times \Omega', \mathcal{A} \times \mathcal{A}')$ is the product space of the two measurable spaces.

2) Product Measure

$(\Omega, \mathcal{A}, \mu)$ and $(\Omega', \mathcal{A}', \nu)$ are 2 measure spaces

$\mathcal{A} \times \mathcal{A}' \equiv \sigma[\mathcal{F}]$

$\mathcal{F} \equiv \{ \sum_{i=1}^{m} A_i \times A_i' \ : m \geq 1 \ , A_i \in \mathcal{A}, \ A_i' \in \mathcal{A}' \}$

and $(\Omega \times \Omega', \mathcal{A} \times \mathcal{A}')$ is the product space

let $\Phi(\sum_{i=1}^{m} A_i \times A_i') \equiv \sum_{i=1}^{m} \mu(A_i) \times \nu(A_i')$

then Φ is a well defined σ-finite meas on $\mathcal{F}$ and Φ extends uniquely to $\mathcal{A} \times \mathcal{A}'$ and it is called the product measure

(d) F is the dist^n funct^n

$$F^{-1}(t) \equiv \inf\{ y \mid F(y) \geq t \} \qquad \text{for } t \in [0,1]$$

is the quantile funct^n

(e) $\{X_n\}$ and $\{Z_n\}$ are 2 seq of r.v. We say they are k-equivalent if $\sum_{n=1}^{\infty} P(X_n \neq Y_n) < \infty$

(f) Let $Y_n = X_n I_{[|X_n| < n]}$ and $\{X_n\}$ is any seq of iid random variables with $E|X_1| < \infty$

then $\{X_n\}$ and $\{Y_n\}$ are K-equivalent.

(II) Statements and examples with proofs.

$\boxed{12 \text{ points}}$ 12/12

(a) 1) State the Radon-Nikodym Theorem for finite measures. (2)

2) Let λ be the Lebesgue measure. (2)
Give an example of a measure μ s.t. $\mu \ll \lambda$ and calculate $\frac{d\mu}{d\lambda}$.

(b) Let $\xi \sim \text{Unif}(0,1)$. Define $X = F^{-1}(\xi)$.
Show that X has distribution F, where
(2) F is a given distribution function.

Hint No need for the full proof of steps of this result. Just state the result you need to make your (one line!) argument.

(c) Let $X_1, \dots, X_n$ be i.i.d. random variables. Under what additional minimal assumption on X_i, $i = \overline{1, n}$, can we conclude that

(1)

$$\sum_{n=1}^{\infty} P(|X_n| \geq n) < \infty \, ?$$

(Just <u>state</u> the assumption; no proof required).

(1) (d) State the weak law of large numbers.

(4) (e) Give the <u>outline</u> of the proof of the WLLN. <u>Only</u> the statement of the main steps is required.

(II) (a)

1) Radon - Nikodyn Theorem

μ and Φ are meas and signed meas, both σ-finite on $(\Omega, \mathcal{A})$ then we say

$\Phi << \mu \iff \exists$ a Z_0, meas, finite valued, unique a.e wrt μ st

$$\Phi(A) = \int_A Z_0 \, d\mu \quad \text{for any } A \in \mathcal{A}$$

$\left[Z_0 \equiv \dfrac{d\Phi}{d\mu} \right.$ is called the R-N derivative of Φ wrt $\left. \mu \right]$

2) λ is the lebesgue measure

Let $X \cong N(\mu, \sigma^2)$ with induced meas P_{μ,σ^2}

then $P_{\mu,\sigma^2} << \lambda$ and

$$P_{\mu,\sigma^2}(A) = \int_A \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\} d\lambda(x)$$

here $$\frac{dP_{\mu,\sigma^2}}{d\lambda}(x) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sigma} \exp\left\{\frac{-1}{2\sigma^2}(x-\mu)^2\right\}$$

(b) $\xi \sim Unif(0,1)$

$X \equiv F^{-1}(\xi)$

<u>Proof</u> : $\{\omega \mid X(\omega) \leq x\} = \{\omega \mid \xi(\omega) \leq F(x)\}$ (we can show)

Now $F_X(x) = P(X \leq x) = P(\{\omega \mid X(\omega) \leq x\})$

$= P(\{\omega \mid \xi(\omega) \leq F(x)\})$

$= P(\xi \leq F(x))$

$= F(x)$ $[\because \xi \sim Unif(0,1)$

$\therefore$ X has df F

To show $\{\omega \mid X(\omega) \leq x\} = \{\omega \mid \xi(\omega) \leq F(x)\}$

Let $\omega_0 \in \Omega$ st $\xi(\omega_0) \leq F(x) \Rightarrow F^{-1}(\xi(\omega_0)) \leq F^{-1}(F(x))$

$\leq x$ $\{F^{-1} \circ F(x) \leq x$

$\Rightarrow X(\omega_0) \leq x$

Let $\omega_0 \in \Omega$ st $X(\omega_0) \leq x \Rightarrow F^{-1}(\xi(\omega_0)) \leq x$

$\Rightarrow \inf\{y \mid F(y) \geq \xi(\omega_0)\} \leq x$

Let $y_0 \equiv \inf\{y \mid F(y) \geq \xi(\omega_0)\}$ then $y_0 \leq x$

$\forall \varepsilon > 0 \ \exists \ y_\varepsilon \in \{y \mid F(y) \geq \xi(\omega_0)\}$ st $y_\varepsilon - \varepsilon \leq y_0$

$\therefore$ we have $\left.\begin{array}{l} y_0 + \varepsilon \geq y_\varepsilon \\ x - y_0 \geq 0 \end{array}\right\} \Rightarrow x + \varepsilon \geq y_\varepsilon$

Since F is df it is $\uparrow$ $\Rightarrow F(x+\varepsilon) \geq F(y_\varepsilon)$

$$F(x+\varepsilon) \geq F(y_\varepsilon) \geq \xi(\omega_0)$$

$$\Rightarrow \quad F(x+\varepsilon) \geq \xi(\omega_0)$$

$$\Rightarrow \quad F(x) \geq \xi(\omega_0) \qquad \{\because \text{ by rt continuity , taking } \varepsilon \to 0$$

$\therefore$ we have shown "$\subseteq$" and "$\supseteq$"

(c) $X_1, X_2 \ldots X_n \ldots$ are iid r.v

If $E|X_1| < \infty$ then $\sum_{n=1}^{\infty} P(|X_n| \geq n) < \infty$

Proof uses the fact that for any r.v X

$$\sum_{n=1}^{\infty} P(|X| \geq n) \leq E|X| \leq \sum_{n=1}^{\infty} P(|X| \geq n) + P(|X| \geq n)$$

So if $E|X| < \infty \Rightarrow \sum_{n=1}^{\infty} P(|X| \geq n) < \infty$

(d) The Weak Law of Large Numbers:

If $X_1, X_2, \ldots X_n \ldots$ are iid r.v with mean μ and $E|X_1| < \infty$ then

$$\bar{X}_n \xrightarrow[n \to \infty]{P} \mu \qquad \text{where } \bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$$

II (e) Proof of WLLN

(i) Let $Y_{nk} = X_k \, I_{[|X_k| < n]}$ and $\bar{Y}_n = \frac{1}{n} \sum_{k=1}^{n} Y_{nk}$

(ii) $\mu_n = E(Y_{nk})$

(iii) Show $\{X_n\}$ and $\{Y_n\}$ are K-equivalent

$\Rightarrow$ If $\bar{Y}_n \longrightarrow$ in P to some Y the $\bar{X}_n \xrightarrow{P}$ to the same Y

(iv) Show $\bar{Y}_n \xrightarrow{\cancel{P}} - \mu_n \xrightarrow{P} 0$ as $n \to \infty$

(v) Show $\mu_n \longrightarrow \mu$ as $n \to \infty$

$\Rightarrow \bar{Y}_n \xrightarrow{P} \mu$ as $n \to \infty$

$\therefore \bar{X}_n \xrightarrow{P} \mu$ (by step (iii))

Exercise 1

(3 points) 3/3

Let $X_1, \dots, X_n$ be i.i.d random variables such that

$$\sum_{i=1}^{n} e^{-X_i} \xrightarrow{a.s.} U,$$ for some given r.v. U.

Find the almost sure limit of

$$\sum_{i=1}^{n} e^{(i - X_i - n)}$$

Exercise 2

(5 points) 5/5

Let $X \geq 0$ be a r.v. with df F.

Show that $EX = \int_0^\infty (1 - F(x))\,dx$.

[Give full explanations at each step and check the conditions of every result you apply in order to receive full credit.]

Excercise 1

$X_1, X_2 \ldots X_n \ldots$ are iid random variables st

$$\sum_{i=1}^{n} e^{-X_i} \xrightarrow{a.s} U \text{ (for some given r.v } U)$$

Now consider $\displaystyle\sum_{i=1}^{n} e^{i-X_i-n} = \sum_{i=1}^{n} \frac{e^{i}\cdot e^{-X_i}}{e^{n}}$

$$= \sum_{k=1}^{n} \frac{b_k Z_k}{b_n}$$

with $b_k \equiv e^{i}$ $\quad Z_k = e^{-X_i}$

Notice $\displaystyle\lim_{k\to\infty} b_k = e^{\infty} = \infty$

$\therefore$ we have a seq $\{b_k\}$ st $b_k \uparrow \infty$

we have $\displaystyle\sum_{k=1}^{n} Z_k \xrightarrow{a.s} U$ (for some U)

$$\therefore \sum_{k=1}^{n} \frac{b_k Z_k}{b_n} \xrightarrow{a.s} 0 \quad \text{(by Kroneckers lemma)}$$

Excercise 2 $X \geq 0$ is a r.v with df F. [say X is def on $(\Omega, \mathcal{A}, P)$)

$$E(X) = \int_{\Omega} X(\omega)\, dP(\omega)$$

$$= \int_{\Omega}\left[\int_0^{X(\omega)} d\lambda(t)\right] dP(\omega)$$ { λ is lebesgue meas

$$= \int_{\Omega}\int_0^{\infty} I_{[0<t<X(\omega)]}\, d\lambda(t)\, dP(\omega)$$

$$= \int_0^{\infty}\int_{\Omega} I_{[0<t<X(\omega)]}\, dP(\omega)\, d\lambda(t)$$ ✓ { by Tonelli since $I_{[0<t<X(\omega)]} \geq 0$ always.

$$= \int_0^{\infty} P(X(\omega) > t)\, d\lambda(t)$$

$$= \int_0^{\infty} (1-F(t))\, dt$$ { by definition of $F(\cdot)$

Note: Here if we use $I_{[0<t\leq X(\omega)]}$ it is also ok since F is rt continuous and $E(0) = 0$

Exercise 3 (7 points) 4/7

a) Let $X_1, \dots, X_n$ be iid with common cdf (5)

$F(x) = (1 - e^{-x})$. 2/5

Define $Y_n \equiv \max_{1 \le i \le n} X_i$. Show that

$P\left(\overline{\lim} \frac{Y_n}{\log n} \le 1 \right) = 0$

$$\overline{\lim} \frac{Y_n}{\log n} \geq 1 \quad \text{a.s.}$$

$P\left(\frac{Y_n}{\log n} \le 1 \text{ i.o.} \right) = 0$

State the results you use. You may need to use the known inequality

$\sum P\left(\frac{Y_n}{\log n} \le 1 \right) < \infty$

$$1 - x \le e^{-x}.$$

b) Let $X_n \sim N(0, n^{\frac{1}{4}})$. Show that (2)

$$\frac{X_n}{n} \xrightarrow{a.s.} 0.$$ 2/2

State the results you are using.

$1 - e^{-x} \le x$

Excercise 3

$X_1, X_2 \ldots X_n \ldots$ are iid with $F(x) = (1-e^{-x})$

$$Y_n \equiv \max_{1\le i\le n} X_i$$

Consider $P(Y_n/\log n \le 1)$

$$= P(Y_n \le \log n)$$

$$= P\left(\max_{1\le i\le n} X_i \le \log n\right)$$

$$\not\le \frac{Var(X_n)}{\log n} \;?$$

Kolmogorov's maximal inequality

If we show $\sum_{n=1}^{\infty} \frac{Var(X_n)}{\log n} < \infty$ then

This is indeed what's needed ... $\Rightarrow \sum_{n=1}^{\infty} P\left(\frac{Y_n}{\log n} \le 1-\varepsilon\right) < \infty$

$$\Rightarrow P\left(\frac{Y_n}{\log n} \le 1-\varepsilon \text{ io}\right) = 0 \quad \{ \text{ Borel Cantelli 1}$$

$$\Rightarrow P\left(\underline{\lim} \frac{Y_n}{\log n} \le 1-\varepsilon\right) = 0$$

$$\Rightarrow \quad \overline{\lim} \frac{Y_n}{\log n} \geq 1 \quad a.s$$

To show $\displaystyle\sum_{n=1}^{\infty} \frac{Var(X_n)}{(\log n)} < \infty$

Consider $\displaystyle\sum_{n=1}^{\infty} \frac{Var(X_n)}{\log n} \leq \sum_{n=1}^{\infty} \frac{E(X_n^2)}{(\log n)} = \sum_{n=1}^{\infty} \frac{2}{\log n} \not< \infty$

No!

Now $\displaystyle E(X_n^2) = \int_0^{\infty} 2t\,(1-F(t))\,dt$

$$= \int_0^{\infty} \underset{u}{2t}\, \underset{dv}{e^{-t} dt}$$

$$= \left\{ 2(-e^{-t})t \Big|_0^{\infty} + 2\int_0^{\infty} e^{-t} dt \right\}$$

$$= 0 + 2\left(-e^{-t}\Big|_0^{\infty}\right)$$

$$= 2e^0$$

$$= 2$$

Excercise 3

(b) $X_n \sim N(0, n^{1/4})$

To show $\frac{X_n}{n} \xrightarrow{a.s} 0$

To show $P\left(\left|\frac{X_n}{n}\right| > \varepsilon \text{ io} \right) = 0$

Consider $P\left(\left| \frac{X_n}{n} \right| > \varepsilon \right) = P\left(|X_n| > n\varepsilon \right)$

$$\leq \frac{Var(X_n)}{n^2 \varepsilon^2}$$ {by Chebyshev's ineq

$$= \frac{n^{1/4}}{n^2 \varepsilon^2}$$

$$= \frac{1}{n^{(2-1/4)} \varepsilon^2}$$

$$\xrightarrow[n\to\infty]{} 0$$

$\Rightarrow \sum_{n=1}^{\infty} P\left(\left|\frac{X_n}{n}\right| > \varepsilon \right) < \infty$ {since the tail $\to 0$

$\Rightarrow P\left(\left|\frac{X_n}{n}\right| > \varepsilon \text{ i.o.} \right) = 0$ {Borel Cantelli 1

$\Rightarrow \frac{X_n}{n} \xrightarrow{a.s} 0$ {by equivalent condn for almost sure convergence.

Excercise 3 (Alternate approach) NOT COMPLETED

To show $P\left(\overline{\lim} \frac{Y_n}{\log n} \leq 1\right) = 0$

$$\iff P\left(\frac{Y_n}{\log n} \leq 1 \ \ io\right) = 0$$

$\therefore$ We can show $\sum_{n=1}^{\infty} P\left(\frac{Y_n}{\log n} \leq 1\right) < \infty$

$$P\left(\frac{Y_n}{\log n} \leq 1\right) = P\left(Y_n \leq \log n\right)$$

$$= P\left(\max_{1 \leq i \leq n} X_i \leq \log n\right)$$

$$= \prod_{i=1}^{n} P\left(X_i \leq \log n\right)$$

$$= \prod_{i=1}^{n} \left(1 - e^{-\log n}\right)$$

$$= \left(1 - n^{-1}\right)^n$$

$$\leq \left(e^{1/n}\right)^n$$

$$= e^{1}$$

Ex 3. MIDTERM

Show $\overline{\lim}\ \frac{Y_n}{\log n} \geq 1$ a.s

To show $P\left(\left[\left(\frac{Y_n}{\log n} \geq 1-\varepsilon\right) \ i.o\right]\right) = 1$

OR $P\left(\frac{Y_n}{\log n} \leq 1-\varepsilon \quad i.o\right) = 0$

To show $\sum_{n=1}^{\infty} P\left(\frac{Y_n}{\log n} \leq 1-\varepsilon\right) < \infty$.

$$P\left(Y_n \leq (1-\varepsilon)\log n\right) = P\left(\max X_n \leq (1-\varepsilon)\log n\right)$$

$$= \prod_{i=1}^{n} P\left(X_i \leq (1-\varepsilon)\log n\right)$$

$$= \left[P\left(X_n \leq (1-\varepsilon)\log n\right)\right]^n$$

$$= \left[1 - e^{-\log n^{(1-\varepsilon)}}\right]^n$$

$$= \left[1 - n^{-(1-\varepsilon)}\right]^n$$

$$\leq \left[e^{-(n^{\varepsilon-1})}\right]^n = \left(e^{-n^{\varepsilon} n^{-1}}\right)^n = e^{-n^{\varepsilon}}$$

$\longrightarrow 0$ as $n \longrightarrow \infty$

MIDTERM

$$\sum_{k=1}^{n} E\left|Y_k I_{[|Y_k| \le 1/\sqrt{n}]}\right|^4$$

$$\le \sum_{k=1}^{n} \left(E\,|Y_k| I_{[|Y_k| \le 1/\sqrt{n}]}\right)^4$$

$$\le \sum_{k=1}^{n} \left(\frac{1}{\sqrt{n}}\right)^4$$

$$= \frac{n}{n^2}$$

$$= \frac{1}{n} \longrightarrow 0$$

Summary of results yielding to the definition of $E(Y|\mathcal{D})$.

- Let Y be a r.v. defined on $(\Omega, \mathcal{A}, P)$.
- Assume $Y \geq 0$. Let $\mathcal{D} \subset \mathcal{A}$ be a σ-field.
- We showed: ① $\nu(D) \equiv \int_D Y\,dP$ is a measure on $\mathcal{D}$, when $D \in \mathcal{D}$.
- We showed: $P_{|\mathcal{D}}$ is a measure on $\mathcal{D}$, where
$$P_{|\mathcal{D}}(D) = P(D), \text{ for all } D \in \mathcal{D}.$$
- Now, from ① and the abs. cont. of the integral (page 42):
$$P_{|\mathcal{D}}(D) = P(D) = 0 \Longrightarrow \nu(D) = 0, \text{ for all } D \in \mathcal{D}.$$
- Thus, by definition, $\nu \ll P_{|\mathcal{D}}$. These two measures are both on $\mathcal{D}$.
- By the R-N Theorem:

There exists a $\mathcal{D}$-measurable function h, unique a.s. $P_{|\mathcal{D}}$ s.t:

$$② \quad \nu(D) \underset{\uparrow}{=} \int_D h\,dP_{|\mathcal{D}} \underset{\uparrow}{=} \int_D h\,dP, \text{ for all } D \in \mathcal{D}.$$

R-N Theorem   definition of $P_{|\mathcal{D}}$

- From ① and ②:

  $\exists\, h$, $\mathcal{D}$-measurable and unique a.s. $P_{|\mathcal{D}}$ s.t.

  ③ $$\int_D Y\,dP = \int_D h\,dP, \qquad \forall D \in \mathcal{D}.$$

- <u>Result</u>: If h is $\mathcal{D}$ measurable and unique a.s $P_{|\mathcal{D}}$ then h is unique a.s. P.

Combining ③ and Result gives:

$\exists\, h$, $\mathcal{D}$-measurable, an unique a.s. P s.t.

$$\int_D Y\,dP = \int_D h\,dP, \qquad \forall D \in \mathcal{D}.$$

- This function h is denoted by $E(Y|\mathcal{D})$ and called the conditional expectation of Y with respect to $\mathcal{D}$, and so:

$$\int_D Y\,dP = \int_D E(Y|\mathcal{D})\,dP, \quad \forall D \in \mathcal{D}.$$