Portmamteau Theorem

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$$\text{Show } F_n \overbrace{\rightarrow}^{d} F \Leftrightarrow E[g_n(x_n)] = \int gdF_n \rightarrow \int gdF = E[g(x)]$$

By **Hally-Bray Theorem**, consider some $(\Omega, \mathbb{A}, P)$. Suppose that g is bdd and is continuous a.s. F. ($g \in C_b$), then

$$E[g_n(x_n)] = \int gdF_n \rightarrow \int gdF = E[g(x)] \tag{1}$$

By converse, since $E[g_n(x_n)] \rightarrow E[g(x)] \Rightarrow F_n \rightarrow F$

$$\text{Show } F_n \overbrace{\rightarrow}^{d} F \Rightarrow E[g_n(x_n)] = \int g dF_n \rightarrow \int g dF = E[g(x)]$$

Notice that for any $\delta(B) = (C_b)^c$. For any B with $P(x \in \delta B) = 0$ by continuous mapping theorem, $E(1_B(x_n)) \rightarrow E(1_B(x))$ as $n \rightarrow \infty$

$F_n(x) = P(x_n \in (-\infty, x])$, with $\delta(-\infty, x] = \{x\}$ so provided $P(X = x) = 0$, then $P(x_n \in (-\infty, x]) \rightarrow P(x \in (-\infty, x])$.

$X \in B \leftrightarrow 1_B$. $X \to 1_B(x)$ is not continuous at δB. Otherwise, $h(x) = 1_B(x)$ is continuous and bounded $\forall x \notin \delta B$

$$X_n \to X \Rightarrow h(x_n) \overbrace{\to}^{a.s} h(x) \text{ and } E[h(x_n)] \overbrace{\to}^{DCT} E[h(x)].$$

$$E[h(x_n)] = E[1_B(x_n)] = P(x_n) \to E[h(x)] = P(x \in B)$$

$$\text{Show } F_n \overbrace{\rightarrow}^{d} F \Leftarrow E[g_n(x_n)] = \int g dF_n \rightarrow \int g dF = E[g(x)]$$

Take $B = (-\infty, x]$, then $\delta B = \{x\}$

$P_n(B) \rightarrow P(B)$ with $P(\delta B) = 0$

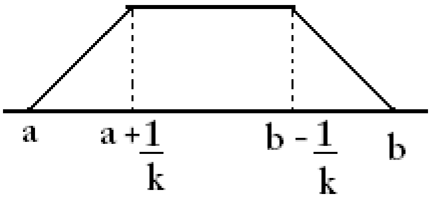
$F_n(x) \rightarrow F(x)$ $\forall x$, $P(\{x, y\}) = 0$, $\forall x$ F has no jumps at x, all $x \in C_b$

For the rest of the proof, define a trapezoidal function of height 1. Let $g_k \in C_b$. Note the $g_k \downarrow 1_{[a,b]}$ as $k \to \infty$.

Since $g_k \leq 1$ and $g_k \downarrow 1_{[a,b]}$, then

$\int g_k dF \downarrow F([a,b]) = F((a,b])$, then

$\limsup F_n(a,b] \leq F(a,b]$

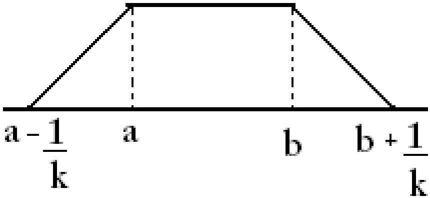


Note the $h_k \uparrow 1_{(a,b)}$

$F_n(a,b] \geq \int h_k dF_n \to \int h_k dF$ by MCT

$\int h_k dF \uparrow F(a,b) = F(a,b]$ as $k \to \infty$.

Thus, $\liminf F_n((a,b]) \geq F(a,b]$



$$\text{Show } E[g_n(x_n)] = \int gdF_n \to \int gdF = E[g(x)] \forall g \in C_b$$
$$\Rightarrow \liminf P_n(B) \geq P(B) \forall \text{ open sets } B$$

Define a function, g, such that satisfies first part, then will show because of the way the function is setup, the function will imply the second part.

Let B be an open set and $F = B^c$. Consider a sequence of functions, $g_n \in C_b$,

$$g_n(x) = \min(1, nd(x, F)) \tag{2}$$

Show that $g_n(x) \uparrow 1(x)$

a. When $x \notin F \Rightarrow$ let $\epsilon = d(x, F) > 0$, then $g_n(x) = \min(1, n\epsilon)$, where $n\epsilon \geq 1$. For n large, $\Rightarrow g_n(x) = 1$
b. When $x \in F$, then $d(x, F) = 0 \Rightarrow g_n(x) = 0$

Therefore,

$$g_n(x) = \begin{cases} 0 & x \notin F = B^c \\ 1 & x \in F = B^c \end{cases}$$

as $n \longrightarrow \infty$

By 9, $P_n(x) \geq \int g_k dF_n \rightarrow \int g_k dF$ as $n \rightarrow \infty$

and

$\liminf P_n(B) \geq \int g_k dF$

by letting $k \rightarrow \infty$ and MCT,

$$\begin{aligned}
P_n(B) &\geq \int g_k dF_n \overbrace{\rightarrow}^{9} \int g_k \uparrow P(B), n \rightarrow \infty \\
\liminf_n P_n(B) &\geq \lim_{k\rightarrow\infty} \int g_k dF \overbrace{=}^{MCT} \int \lim_{k\rightarrow\infty} g_k dF = \int g dF = P(B) \\
\liminf_n P_n(B) &\geq \int 1_B dF = P(B) \qquad (3)
\end{aligned}$$

$$\text{Show } E[g_n(x_n)] = \int g dF_n \to \int g dF = E[g(x)] \forall g \in C_b$$
$$\Rightarrow \limsup P_n(B) \leq P(B) \forall \text{ closed sets } B$$

$$\begin{aligned} P(B) &= \int 1_B dF_n \leq \int g_k dF_n \overbrace{\to}^{9} \int g_k dF \downarrow P(B^c) \\ \limsup_n P_n(B) &\leq \lim_{k\to\infty} \int g_k dF_n \overbrace{=}^{MCT} \int \lim_{k\to\infty} g_k dF = \int g DF = P(B) \end{aligned}$$

$$\begin{aligned} \text{Show } \limsup P_n(B) &\leq P(B)\forall \text{ closed sets } B \\ \Rightarrow \liminf P_n(B) &\geq P(B)\forall \text{ open sets } B \end{aligned}$$

$$\begin{aligned} \limsup P_n(B) &\leq P(B)\forall \text{ closed sets } B \\ 1 - \limsup P_n(B) &\geq P(B^c) \\ 1 + \liminf(-P_n(B)) &\geq P(B^c) \\ \liminf(1 - P_n(B)) &\geq P(B^c) \\ \liminf(P_n(B^c)) &\geq P(B^C) \end{aligned} \tag{4}$$

$$\begin{aligned} \text{Show } \liminf P_n(B) &\geq P(B) \forall \text{ open sets } B \\ \Rightarrow \limsup P_n(B) &\leq P(B) \forall \text{ closed sets } B \end{aligned}$$

$$\begin{aligned} \liminf P_n(B) &\geq P(B) \forall \text{ open sets } B \\ 1 - \liminf P_n(B) &\leq P(B^c) \\ 1 + \limsup(-P_n(B)) &\leq P(B^c) \\ \limsup(1 - P_n(B)) &\leq P(B^c) \\ \limsup P_n(B^c) &\leq P(B^c) \qquad (5) \end{aligned}$$

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\begin{aligned}
\text{Show } \liminf P_n(B) &\geq P(B) \forall \text{ open sets } B \\
\text{and } \limsup P_n(B) &\leq P(B) \forall \text{ closed sets } B \\
\Rightarrow \lim P_n(B) &= P(B)
\end{aligned}
$$

Define the following:

a. B^0 is the interior of B, where $B^0 = \cup\{U \subset B, \text{ s.t. } U \text{ is open}\}$, (Union of all open subsets of B.

b. $\bar{B}$ is the closure of B, where $\bar{B} = \bigcap\{B \subset C, \text{ s.t. } C \text{ is closed}\}$, (Intersection of all closed subsets of B.

$$P(B) \leq \liminf P_n(B) \leq \limsup P_n(B) \leq P(B)$$

$\forall$ $P-$continuity sets B, $(P(\delta B) = 0)$, then the boundary B, $\delta B = \bar{B}/B^0$
Since $P(\delta B) = 0$, then $P(\bar{B}) = P(B^0) = P(B) \rightarrow \lim P_n(B) = P(B)$

$$\text{Show } \lim P_n(B) = P(B) \to F_n \overbrace{\to}^{d} F$$

For $\lim P_n(B) = P(B) \ \forall \ P-$ continuity sets B, , $(P(\delta B) = 0)$, where $B \subset \Omega = \mathbb{R}$.
Let $B = (-\infty, x] \to \delta(-\infty, x] = \{x\}$. Then $P(-\infty, x]) = 0 \ \forall$ continuity points $x \in \Omega$

$$F_n \overbrace{\to}^{d} F \Rightarrow F_n((-\infty, x]) \to F((-\infty, x])$$