Portmamteau Theorem

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1. Show $7 \longrightarrow 9$
2. Show $7 \longrightarrow 9$
3. Show $9 \longrightarrow 12$
4. Show $9 \longrightarrow 11$
5. Show $11 \longrightarrow 12$
6. Show $12 \longrightarrow 11$
7. Show 11 and $12 \longrightarrow 13$
8. Show $13 \longrightarrow 9$

$$F_n \overbrace{\longrightarrow}^{d} F \Rightarrow E[g_n(x_n)] = \int g dF_n \rightarrow \int g dF = E[g(x)] \tag{1}$$

By **Hally-Bray Theorem**, consider some $(\Omega, \mathbb{A}, P)$. Suppose that g is bdd and is continuous a.s. F. $(g \in C_b)$, then

$$E[g_n(x_n)] = \int g dF_n \rightarrow \int g dF = E[g(x)] \tag{2}$$

By converse, since $E[g_n(x_n)] \rightarrow E[g(x)] \Rightarrow F_n \rightarrow F$

$\longleftarrow$: Notice that for any $\delta(B) = (C_b)^c$. For any B with $P(x \in \delta B) = 0$ by continuous mapping theorem, $E(1_B(x_n)) \longrightarrow E(1_B(x))$ as $n \longrightarrow \infty$

$F_n(x) = P(x_n \in (-\infty, x])$, with $\delta(-\infty, x] = \{x\}$ so provided $P(X = x) = 0$, then $P(x_n \in (-\infty, x]) \rightarrow P(x \in (-\infty, x])$.

$x \in B \leftrightarrow 1_B$. $x \rightarrow 1_B(x)$ is not continuous at δB. Otherwise, $h(x) = 1_B(x)$ is continuous and bounded $\forall x \notin \delta B$

$x_n \rightarrow x \Rightarrow h(x_n^{\#}) \overbrace{\rightarrow}^{a.s} h(x^{\#})$ and $E[h(x_n^{\#})] \overbrace{\rightarrow}^{DCT} E[h(x^{\#})]$. So $E[h(x_n)] \rightarrow E[h(x)]$

$E[h(x_n)] = E[1_B(x_n)] = P(x_n) \rightarrow E[h(x)] = P(x \in B)$

$\longrightarrow$: Take $B = (-\infty, x]$, then $\delta B = \{x\}$

$P_n(B) \rightarrow P(B) \quad \forall B \quad \text{with } P(\delta B) = 0$

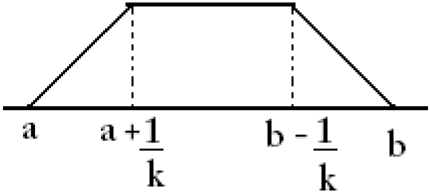
$F_n(x) \rightarrow F(x)$ $\forall x$, $P(\{x\}) = 0$, $\forall x$ F has no jumps at x, $\rightarrow$ all $x \in C_b$

Define a trapezoidal function of height 1. Let $g_k \in C_b$. Note the $g_k \downarrow 1_{[a,b]}$ as $k \rightarrow \infty$.

Since $g_k \leq 1$ and $g_k \downarrow 1_{[a,b]}$, then

$\int g_k dF \downarrow F([a,b]) = F((a,b])$, then

$\limsup F_n(a,b] \leq F(a,b]$

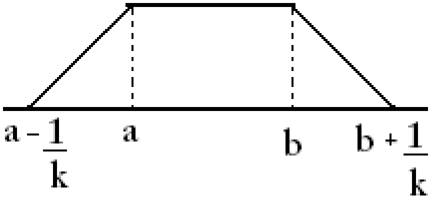


Note the $h_k \uparrow 1_{(a,b)}$

$F_n(a,b] \geq \int h_k dF_n \rightarrow \int h_k dF$ by MCT

$\int h_k dF \uparrow F(a,b) = F(a,b]$ as $k \rightarrow \infty$.

Thus, $\liminf F_n((a,b]) \geq F(a,b]$



Show that $E[g_n(x_n)] = \int gdF_n \rightarrow \int gdF = E[g(x)]$ $\forall$ $g \in C_b \Rightarrow \liminf P_n(B) \geq P(B)$ $\forall$ open sets B

Define a function, g, such that satisfies first part, then will show because of the way the function is setup, the function will imply the second part.

Let B be an open set and $F = B^c$. Consider a sequence of functions, $g_n \in C_b$,

$$g_n(x) = \min(1, nd(x, F)) \tag{3}$$

Show that $g_n(x) \uparrow 1(x)$

a. When $x \notin F \Rightarrow$ let $\epsilon = d(x, F) > 0$, then $g_n(x) = \min(1, n\epsilon)$, where $n\epsilon \geq 1$. For n large, $\Rightarrow g_n(x) = 1$

b. When $x \in F$, then $d(x, F) = 0 \Rightarrow g_n(x) = 0$

Therefore,

$$g_n(x) = \begin{cases} 0 & x \notin F = B^c \\ 1 & x \in F = B^c \end{cases}$$

as $n \longrightarrow \infty$

By 9, $P_n(x) \geq \int g_k dF_n \longrightarrow \int g_k dF$ as $n \longrightarrow \infty$

and $\liminf P_n(B) \geq \int g_k dF$

by letting $k \longrightarrow \infty$ and MCT,

$$
\begin{aligned}
P_n(B) &\geq \int g_k dF_n \overbrace{\longrightarrow}^{9} \int g_k dF \uparrow P(B), n \longrightarrow \infty \\
\liminf_n P_n(B) &\geq \lim_{k\to\infty} \int g_k dF \overbrace{=}^{MCT} \int \lim_{k\to\infty} g_k dF = \int g dF = P(B) \\
\liminf_n P_n(B) &\geq \int 1_B dF = P(B) \qquad (4)
\end{aligned}
$$

Show that $E[g_n(x_n)] = \int g dF_n \to \int g dF = E[g(x)] \quad \forall$
$g \in C_b \Rightarrow \limsup P_n(B) \le P(B) \quad \forall$ closed sets B

$$P(B) = \int 1_B dF_n \le \int g_k dF_n \overbrace{\to}^{9} \int g_k dF \downarrow P(B^c)$$

$$\limsup_n P_n(B) \le \lim_{k\to\infty} \int g_k dF_n \overbrace{=}^{MCT} \int \lim_{k\to\infty} g_k dF = \int g dF = P(B)$$

$$
\begin{aligned}
\text{Given } \limsup P_n(B) &\le P(B) \ \forall \text{ closed sets } B \\
1 - \limsup P_n(B) &\ge P(B^c) \\
1 + \liminf(-P_n(B)) &\ge P(B^c) \\
\liminf(1 - P_n(B)) &\ge P(B^c) \\
\liminf P_n(B^c) &\ge P(B^C) \qquad (5)
\end{aligned}
$$

$$\begin{aligned} \text{Given } \liminf P_n(B) &\geq P(B) \ \forall \text{ open sets } B \\ 1 - \liminf P_n(B) &\leq P(B^c) \\ 1 + \limsup(-P_n(B)) &\leq P(B^c) \\ \limsup(1 - P_n(B)) &\leq P(B^c) \\ \limsup P_n(B^c) &\leq P(B^c) \qquad (6) \end{aligned}$$

Define the following:

a. B^0 is the interior of B, where $B^0 = \cup\{U \subset B, \text{ s.t. } U \text{ is open}\}$, (Union of all open subsets of B.)

b. $\bar{B}$ is the closure of B, where $\bar{B} = \bigcap\{B \subset C, \text{ s.t. } C \text{ is closed}\}$, (Union of all open subsets of B.)

$$P(B) \leq \liminf P_n(B) \leq \limsup P_n(B) \leq P(B)$$

$\forall$ $P-$continuity sets B, $(P(\delta B) = 0)$, then the boundary B, $\delta B = \bar{B}/B^0$

Then $P(\delta B) = 0$, then $P(\bar{B}) = P(B^0) = P(B) \to \lim P_n(B) = P(B)$

For $\lim P_n(B) = P(B) \;\forall\; P-$ continuity sets B, , $(P(\delta B) = 0)$, where $B \subset \Omega = \mathbb{R}$.

Let $B = (-\infty, x] \rightarrow \delta(-\infty, x] = \{x\}$. Then $P((-\infty, x]) = 0 \;\forall$ continuity points $x \in \Omega$

$$F_n \rightarrow F \Rightarrow F_n((-\infty, x]) \rightarrow F((-\infty, x])$$