Applied Nonparametrics
STA 4502/5507

Yiyuan She

Department of Statistics, Florida State University

Fall 2009

Possible Course Flow

- **Estimating success probabilities**
- Single location: estimates, tests, intervals
- Two locations: testing, estimating differences between locations
- Scale comparisons
- Multiple locations and factors
- Independence
- Nonparametric regression
- Other topics ...

Binomial Test

- Observe outcomes of n trials
- Assumptions
  - Independent
  - Success / failure
  - Identical
- Test

$$H_0 : p = p_0$$

- p_0 a specified value in $(0, 1)$
- Use a test statistic

$$B = \text{ number of successes}$$

Binomial Test

- One-sided alternative

$$H_1 : p > p_0$$

- For a specified level of significance α, get critical value b_α
- $\alpha = P(\text{reject } H_0 | H_0 \text{ true}) = \text{Type I error}$
- Table A.2 in NSM or `pbinom` in R
- Reject H_0 if $B \geq b_\alpha$

Binomial Test

- One-sided alternative

$$H_1 : p < p_0$$

- For a specified level of significance α, get critical value c_α
- Reject H_0 if $B \leq c_\alpha$
- If $p = 1/2$, symmetry gives $c_\alpha = n - b_\alpha$

Binomial Test

- Two-sided alternative

$$H_1 : p \neq p_0$$

- Reject H_0 if $B \geq b_{\alpha_1}$ or $B \leq c_{\alpha_2}$
- $\alpha = \alpha_1 + \alpha_2$
- Usually, $\alpha_1 = \alpha_2 = \alpha/2$

Binomial Test

- `pbinom(q, size, prob, lower.tail=T)`
- `pbinom(2, 5, 0.4, lower.tail=T)` is $P(B \leq 2)$ where there are $n = 5$ trials and success probability is 0.4
- `pbinom(2, 5, 0.4, lower.tail=F)` is $P(B > 2) = 1 - P(B \leq 2)$
- Note $P(B > 2) = P(B \geq 3)$
- Default is `lower.tail=T`

Binomial Test

- `?pbinom` for documentation
- Density, distribution function, quantile function and random generation for the binomial distribution with parameters size and prob.
  - `dbinom(x, size, prob, log = FALSE)`
  - `pbinom(q, size, prob, lower.tail = TRUE, log.p = FALSE)`
  - `qbinom(p, size, prob, lower.tail = TRUE, log.p = FALSE)`
  - `rbinom(n, size, prob)`

Binomial Test

- Example 2.1
- 7 gaps examined
- 6 were ‘successes’ (more dead branches than lives ones)
- Success probability is 0.15 in previous studies
- Is success probability in this study greater than others?

Binomial Test

$$H_0 : p = 0.15$$
$$H_1 : p > 0.15$$

- $B = 6, n = 7, p_0 = 0.15$
- Set $\alpha = 0.0121$
- Critical-value method:

$$P(R|H_0) = \alpha$$

  or $P(S \geq b_\alpha) = \alpha$, where $S \sim Bin(7, .15)$
- Table A.2 $\Rightarrow b_\alpha = 4$ (`qbinom(1-.01210317,7,.15)+1`)
- Reject null since $B \geq b_\alpha = 4$
- p-value?

Binomial Test

- p-value: smallest significance level at which we would reject the null based on the given data
- Already know we reject the null if $\alpha = 0.0121$
- Smaller α?
- Table A.2, $\alpha = 0.0012, b_\alpha = 5 \Rightarrow$ reject null
- Table A.2, $\alpha = 0.0001, b_\alpha = 6 \Rightarrow$ reject null
- Table A.2, $\alpha = 0.0000, b_\alpha = 7 \Rightarrow$ do not reject null

Binomial Test

- If null is true, what is $P(B \geq 6)$?
- Table A.2 $\Rightarrow$ 0.0001
- $P(B \geq 6) = P(B > 5) =$ `pbinom(5, 7, 0.15, lower.tail=F)`
- $P(B \geq 6) = 1 - P(B \leq 5) =$ `1-pbinom(5, 7, 0.15)`
- `6.948281e-05` $= 0.00006948281 \approx 0.0001$

Binomial Test

- p-value method is more informative
- p-val: 'true' Type-I error for rejecting the null
- If p-value is extremely small, one can reject H_0 at basically no risk

Binomial Test

- `binom.test`
- `?binom.test`
- `binom.test(x, n, p = 0.5, alternative = c("two.sided", "less", "greater"), conf.level = 0.95)`
- `binom.test(6, 7, .15, "greater")`

Binomial Test

- parametric
- $\longrightarrow$ many nonparametric methods
- Those three assumptions (i.i.d. + Bernoulli) lead to Binomial problem
- In NSM, it is said to be a distribution-free test (Comment 2)

Large Sample Test

- B is asymptotically normal (CLT)
- Previous nonparametric test is exact
- This test will be approximate (OK for large samples)
- Under H_0:
  - $E(B) = np_0$
  - $\text{var}(B) = np_0(1 - p_0)$
- Standardize B

$$B^* = \frac{B - np_0}{\sqrt{np_0(1 - p_0)}}$$

- B^* approximately normal(0, 1)
- Critical values are now z_α (A.1)

Large Sample Test

- Example 2.2
- 50 taste tests
- Success correctly identified flavor
- 25 successes
- Is probability of success greater than $1/3$?
- (iid assumption valid? panelists: independent & homogenous judges)

Large Sample Test

$$H_0 : p = 1/3$$
$$H_1 : p > 1/3$$

- $B = 25, n = 50, p_0 = 1/3$
- Set $\alpha = 0.05$
- Table A.1 $\Rightarrow z_\alpha = 1.645$
- `qnorm(.95, 0, 1) = 1.644854`
- Reject H_0 since $B^* = 2.5 \geq z_\alpha$
- p-value: `pnorm(2.5, 0, 1, lower.tail=F) = 0.006209665`

Power

- $\alpha = P(\text{reject } H_0 | H_0 \text{ true}) = \text{Type I error}$
- $\beta = P(\text{do not reject } H_0 | H_0 \text{ false}) = \text{Type II error}$
- Power $= 1 - \beta = P(\text{reject } H_0 | H_0 \text{ false})$
- Consistent test has power $\to 1$ as $n \to \infty$
- Power varies based on conditional event: H_0 false
- Many ways for H_0 to be false, one way for it to be true

Power

$$H_0 : p = 0.4$$
$$H_1 : p > 0.4$$

- $n = 8$
- $\alpha = 0.0085 \Rightarrow$ reject null if $B \geq 7$
- Suppose $p = 0.5$, the null is false
- Power $= P(B \geq 7 | p = 0.5) = 0.0352$ (A.2)
- $\beta = 1-$ power $= 0.9648$ (R: `1-pbinom(6, 8, .5)`)

Power

- Try different α
- $\alpha = 0.0498 \Rightarrow$ reject null if $B \geq 6$
- Suppose $p = 0.5$, the null is false
- Power $= P(B \geq 6|p = 0.5) = 0.1445$ (A.2)
- $\beta = 1-$ power $= 0.8554$ ($\gg \alpha \approx 0.05$)
- $\alpha \uparrow, \beta \downarrow$

Power

- $n = 50, \alpha = 0.05$ (taste test)
- Suppose $p = 0.6$, the null is false

$$\frac{B - (50)(0.6)}{\sqrt{(50)(0.6)(0.4)}}$$

is (asymptotically) normal(0, 1)

$$B^* = \frac{B - (50)(1/3)}{\sqrt{(50)(1/3)(2/3)}}$$

is (asymptotically) normal, but not standard

Power

$$
\begin{aligned}
\text{Power} &= P(B^* > 1.645 | p = 0.6) \\
&= P_{0.6}\left(\frac{B - (50)(1/3)}{\sqrt{(50)(1/3)(2/3)}} > 1.645\right) \\
&= P_{0.6}\left(B > \underbrace{(50)(1/3) + (1.645)\sqrt{(50)(1/3)(2/3)}}_{x}\right) \\
&= P_{0.6}\left(\frac{B - (50)(0.6)}{\sqrt{(50)(0.6)(0.4)}} > \frac{x - (50)(0.6)}{\sqrt{(50)(0.6)(0.4)}}\right) \\
&\approx P(Z > -2.27) = 0.9884
\end{aligned}
$$

Note that n is large here

Estimating p

- Point estimate: $\hat{p} = B/n$
- $E(\hat{p}) = p$ (unbiased)
- $\text{var}(\hat{p}) = \frac{p(1-p)}{n}$, $\text{sd}(\hat{p}) = \sqrt{\frac{p(1-p)}{n}}$
- Don't know p
- Standard error:
$$\hat{\text{sd}}(\hat{p}) = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

Estimating p

- Reducing error of $\hat{p}$
- Want $|p - \hat{p}| < D$ with probability $1 - \alpha$
- Increase n
- Problem: How much? (Sample size determination)
- Asymptotically, we have

$$P\left(-z_{\alpha/2} < \frac{\hat{p} - p}{\sqrt{p(1-p)/n}} < z_{\alpha/2}\right) = 1 - \alpha$$

Estimating p

- Or,

$$z_{\alpha/2} = \frac{D}{\sqrt{p(1-p)/n}}$$

- Choose n such that

$$n = \frac{p(1-p)z_{\alpha/2}^2}{D^2}$$

- Don't know p, substitute in worst case, $p = 1/2$

$$n = \frac{z_{\alpha/2}^2}{4D^2}$$

Estimating p

- Interval estimate

$$P(p_L(\alpha) < p < p_U(\alpha)) \geq 1 - \alpha$$

- Use Table A.3 (not necessarily symmetric; exact) (Clopper-Pearson)
- Asymptotically,

$$P\left(-z_{\alpha/2} < \frac{p - \hat{p}}{\sqrt{p(1-p)/n}} < z_{\alpha/2}\right) = 1 - \alpha$$

- This gives

$$p_L(\alpha) = \hat{p} - z_{\alpha/2}\sqrt{\hat{p}(1-\hat{p})/n}$$
$$p_U(\alpha) = \hat{p} + z_{\alpha/2}\sqrt{\hat{p}(1-\hat{p})/n}$$

- Approximate, may be outside (0, 1)

Estimating p

- `binom.test(6, 7, .15, "two.sided")`
  `[0.4212768, 0.9963897]`
- `binom.test(6, 7, .15, "greater", conf.level=.90)`
  `[0.5474351, 1.0000000]`