2.9.15 p.52.

(a).

μ_1 Σ_{11} Σ_{12}

$$\begin{bmatrix} x_1 \\ \hline x_2 \\ x_3 \end{bmatrix} \sim N_3\left(\begin{bmatrix} 5 \\ \hline 10 \\ 2 \end{bmatrix}, \left[\begin{array}{c|cc} 4 & 1 & -1 \\ \hline 1 & 2 & 0 \\ -1 & 0 & 1 \end{array}\right] \right).$$

μ_2 Σ_{21} Σ_{22}.

$$\underline{x}_2 = \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} \Big| x_1 \sim N_2\Big(\mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(x_1 - \mu_1),$$

$$\Sigma_{2.1} = \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12} \Big).$$

$$\sim N_2\left(\begin{pmatrix} 10 \\ 2 \end{pmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \times \frac{1}{4} \times (x_1 - 5), \right.$$

$$\left. \Sigma_{2.1} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot \frac{1}{4} \begin{bmatrix} 1 & -1 \end{bmatrix} \right) //.$$

$$= N_2\left(\begin{pmatrix} 10 \\ 2 \end{pmatrix} + \begin{bmatrix} (x_1 - 5)/4 \\ -(x_1 - 5)/4 \end{bmatrix}, \Sigma_{1.2} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right.$$

2.9.15'. (b).

$$\begin{bmatrix} x_1+x_2 \\ x_1-x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = C\underline{x}.$$

$$C\underline{x} \sim N_3(\underline{C}.\underline{\mu},\ C\Sigma C').$$

$$= N_3\left(\overset{C\mu}{\begin{bmatrix} 15 \\ -5 \\ 2 \end{bmatrix}},\ \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

$$= N_3\left(\begin{pmatrix} 15 \\ -5 \\ 2 \end{pmatrix},\ \begin{bmatrix} 8 & 2 & -1 \\ 2 & 4 & -1 \\ -1 & -1 & 1 \end{bmatrix} \right).$$

(c)

$$A\underline{x} \sim N_2\left(A\underline{\mu},\ A\Sigma A'\right).$$

$$= N_2\left[\left(\begin{matrix} 1/2 & -1 & 1/2 \\ -1/2 & 0 & 1/2 \end{matrix} \right] \begin{bmatrix} 5 \\ 10 \\ 2 \end{bmatrix},\right.$$

$$\left. \begin{bmatrix} 1/2 & -1 & 1/2 \\ -1/2 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 4 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 \\ -1 & 0 \\ 1/2 & 1/2 \end{bmatrix} \right).$$

$$= N_2\left[\begin{bmatrix} -6.5 \\ -1.5 \end{bmatrix},\ \begin{bmatrix} 1.75, & -0.25 \\ -0.25, & 1.75 \end{bmatrix} \right).$$

$$= N_2\left(\begin{bmatrix}8\\2\end{bmatrix} + \begin{bmatrix}-1\\-1\end{bmatrix} 1^{-1}(x_3-2),\right.$$

$$\Sigma_{1.2} = \begin{bmatrix}8 & 2\\2 & 4\end{bmatrix} - \begin{bmatrix}-1\\-1\end{bmatrix} 1^{-1} [-1, -1]$$

$$= \begin{bmatrix}8 & 2\\2 & 4\end{bmatrix} - \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$$

$$\left. = \begin{bmatrix}7 & 1\\1 & 3\end{bmatrix} \right).$$

(c). $A\underline{x} \sim N_2(A\underline{\mu}, A\Sigma A')$.

$$= N_2\left(\begin{bmatrix}1/2 & -1 & 1/2\\-1/2 & 0 & 1/2\end{bmatrix}\begin{bmatrix}5\\10\\2\end{bmatrix},\right.$$

$$\left.\begin{bmatrix}1/2 & -1 & 1/2\\-1/2 & 0 & 1/2\end{bmatrix}\begin{bmatrix}4 & 1 & -1\\1 & 2 & 0\\-1 & 0 & 1\end{bmatrix}\begin{bmatrix}1/2 & -1/2\\-1 & 0\\1/2 & 1/2\end{bmatrix}\right)$$

$$= N_2\left(\begin{bmatrix}-6.5\\-1.5\end{bmatrix}, \begin{bmatrix}1.75, & -0.25\\-0.25, & 1.75\end{bmatrix}\right).$$

2.9.27.

$x_1 - x_2 = [1 \ -1 \ 0]\underline{x} = a'\underline{x}$.

$\sim N_1(a'\mu, a'\Sigma a)$

$\sim N\left(\mu_1 - \mu_2, (1 -1, 0)\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & 1 & 1/3 \\ 1/3 & 1/3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right)$

$\equiv N\left(\mu_1 - \mu_2, (\frac{1}{2} \ -\frac{1}{2} \ 0)\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right)$.

$\equiv N(\mu_1 - \mu_2, 1)$

//.

(b). $x_1 + x_2 + x_3 = (1, 1, 1)\underline{x}$

$\sim N(a'\mu, a'\Sigma a)$.

$\equiv N\left(\mu_1 + \mu_2 + \mu_3, (1 \ 1 \ 1)\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & 1 & 1/3 \\ 1/3 & \frac{1}{3} & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$.

$\equiv N\left(\mu_1 + \mu_2 + \mu_3, (\frac{11}{6}, \frac{11}{6}, \frac{5}{3})\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$

$\equiv N\left(\mu_1 + \mu_2 + \mu_3, \frac{32}{6}\right)$. //.

(c) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \sim N_2\left(\underbrace{\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}}_{\underline{\mu}_1}, \underbrace{\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}}_{\Sigma_{11}} \right),$

(d). $$\begin{bmatrix} x_1 \\ x_2 \\ \hline x_3 \end{bmatrix} \begin{matrix} \leftarrow \underline{x}_1 \\ \\ \leftarrow \underline{x}_2 \end{matrix} \sim N\left(\begin{bmatrix} \mu_1 \\ \mu_2 \\ \hline \mu_3 \end{bmatrix} \begin{matrix} \leftarrow \underline{\mu}_1 \\ \\ \leftarrow \underline{\mu}_2 \end{matrix}, \begin{array}{c} \begin{array}{cc} \Sigma_{11} & \Sigma_{12} \end{array} \\ \left[\begin{array}{cc|c} 1 & 1/2 & 1/3 \\ 1/2 & 1 & 1/3 \\ \hline 1/3 & 1/3 & 1 \end{array} \right] \\ \begin{array}{cc} \Sigma_{21} & \Sigma_{22} \end{array} \end{array} \right).$$

$$\underline{x}_1 | \underline{x}_2 \sim N_2 \left(\underline{\mu}_1 + \Sigma_{12} \Sigma_{22}^{-1} (\underline{x}_2 - \underline{\mu}_2), \right.$$
$$\left. \Sigma_{1.2} = \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21} \right).$$

$$= \left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} + \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix} (1)^{-1} (x_3 - \mu_3), \right.$$
$$\left. \Sigma_{1.2} = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix} - \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix} (1)^{-1} \left(\tfrac{1}{3} \cdot \tfrac{1}{3} \right) \right)$$

$$= \left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \frac{1}{3} \cdot \begin{bmatrix} x_3 - \mu_3 \\ x_3 - \mu_3 \end{bmatrix}, \right.$$
$$\left. \Sigma_{1.2} = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix} - \frac{1}{9} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. \right)$$

3 (6)

(a). $Cov(X_1, X_2) = -2 \Rightarrow X_1 \not\perp X_2$.

(b). $\begin{bmatrix} X_2 \\ X_3 \end{bmatrix} \sim N_2$. and $Cov(X_2, X_3) = 0$

$\Rightarrow X_2 \perp X_3$.

(c). $Cov\left(\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, X_3\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

and $\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \sim N_3$.

$\therefore \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \perp X_3$.

(d.) $Cov(X_2, X_2 - 2.5X_1 - X_3)$.

$= \sigma_{22} - 2.5\sigma_{21} - \sigma_{13}$

$= 5 - 2.5 \times (-2) - (0)$.

$= 10 \neq 0 \quad \therefore X_2 \not\perp X_2 - 2.5X_1 - X_3$.