**THE UNIVERSITY OF CHICAGO**
**Graduate School of Business**
Business 41912, Spring Quarter 2002, Mr. Ruey S. Tsay

## Solutions to Midterm

1. Make use of properities of multivariate normal distribution and that of matrix to answer this question.

   (a) What is the distribution of $\boldsymbol{X} = (Z_1, Z_2)'$?

   Answer: $\boldsymbol{X} \sim N_2\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right)$.

   (b) What is the conditional distribution of $Z_1$ given that $Z_3 = 3$?

   Answer: $Z_1$ and $Z_3$ are independent so that $Z_1|Z_3 \sim N(3, 2)$.

   (c) Find a linear combination $y = \boldsymbol{C}'\boldsymbol{X}$ such that $\boldsymbol{C}'\boldsymbol{C} = 1$ and $\text{Var}(y)$ is as large as possible. What is the value of the resulting $\text{Var}(y)$?

   Answer: The eigenvalues of the covariance matrix of $\boldsymbol{X}$ are 3 and 1. Specifically, let $\boldsymbol{A}$ be the covariance matrix of $\boldsymbol{X}$. Solve the equation $|\boldsymbol{A} - \lambda\boldsymbol{I}| = 0$ to obtain the two eigenvalues. The eigenvectors are $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})'$ and $(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})'$, respectively. Consequently, $\boldsymbol{C} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})'$ and $\text{Var}(y) = 3$.

   (d) What are the eigenvalues of $\boldsymbol{\Sigma}$?

   Answer: From the block diagonal structure of $\boldsymbol{\Sigma}$. The eigenvalues are 3, 1, 2.

   (e) Find the squared root matrix of $\boldsymbol{\Sigma}$.

   Answer: Again, from the block diagonal structure of $\boldsymbol{\Sigma}$, we can find the squared root of the covariance matrix $\boldsymbol{A}$ of $\boldsymbol{X}$ first, which is

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}.$$

   From the zero covariance matrix between $\boldsymbol{X}$ and $Z_3$, the last eigenvector is $(0, 0, 1)'$. Therefore, the squared root of $\boldsymbol{\Sigma}$ is

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}+1}{2} & \frac{\sqrt{3}-1}{2} & 0 \\ \frac{\sqrt{3}-1}{2} & \frac{\sqrt{3}+1}{2} & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}.$$

   (f) Consider the linear regression model

$$Z_1 = \beta_0 + \beta_1 Z_2 + \beta_2 Z_3 + \epsilon,$$

where $E(\epsilon) = 0$ and $\epsilon$ is uncorrelated with $Z_3$ and $Z_2$. Find the values of $\beta_0$, $\beta_1$ and $\beta_2$.

Answer: This question can be answered in several ways. I show two approaches.

**Simple approach**: Note that $Z_1$ and $Z_3$ are independent so that $\beta_2 = 0$ and the regression becomes a simple linear regression. Consequently, $\beta_1 = \frac{\text{Cov}(Z_1,Z_2)}{\text{Var}(Z_2)} = 1/2 = 0.5$ and $\beta_0 = E(Z_1) - \beta_1 E(Z_2) = 3 - 0.5 \times 2 = 2$.

**Tedious approach**: Use the following properties: (a) $E(XY) = \text{Cov}(X,Y) + E(X)E(Y)$. (b) $E(X^2) = \text{Var}(X) + [E(X)]^2$. (c) $Z_3$ is independent of $Z_1$ and $Z_2$.

Taking the expectation of the model, we have $E(Z_1) = \beta_0 + \beta_1 E(Z_2) + \beta_2 E(Z_3)$. Multiplying the equation of $Z_2$ and taking expectation, we have $E(Z_1 Z_2) = \beta_0 + \beta_1 E(Z_2^2) + \beta_2 E(Z_2)E(Z_3)$. Similarly, multiplying the model by $Z_3$ and taking expectation, we have $E(Z_1)E(Z_3) = \beta_0 + \beta_1 E(Z_2)E(Z_3) + \beta_2 E(Z_3^2)$. Solve the three lienar equations, we obtain $\beta_0 = 2$, $\beta_1 = 0.5$ and $\beta_2 = 0$.

2. Use results of Chapter 5.

   (a) Construct at 95% confidence interval of the stiffness. You may use $t_{29}(0.975) = 2.045$.
   Answer: $18.605 \pm 2.045 \times \sqrt{12.4055/30}$, i.e. (17.29,19.92).

   (b) Test the null hypothesis $H_o : \boldsymbol{\mu} = (20, 10)'$ vs $H_a : \boldsymbol{\mu} \neq (20, 10)'$. You may use $F_{2,28}(0.95) = 3.34$ to perform the test. Draw your conclusion.
   Answer: Use Hotelling $T^2$ in Eq. (5.4) on page 211,

$$T^2 = n(\boldsymbol{x} - \boldsymbol{\mu}_0)'\boldsymbol{S}^{-1}(\boldsymbol{x} - \boldsymbol{\mu}_0) = 23.637.$$

   The critical value is $\frac{(n-1)p}{n-p} F_{p,n-p}(0.95) = 6.919$. Thus, reject the null hypothesis.

3. Use results of Chapter 6.

   (a) Write $\bar{\boldsymbol{x}} = (\bar{x}_1, \bar{x}_2, \bar{x}_3)'$. What is the variance of $\bar{x}_1 - \bar{x}_2$?
   Answer: $\text{Var}(\bar{x}_1 - \bar{x}_2) = \text{Var}(\bar{x}_1) + \text{Var}(\bar{x}_2)$ -2 $\text{Cov}(\bar{x}_1, \bar{x}_2) = \frac{101+80-2\times 63}{40} = 1.375$.

   (b) Construct a $\boldsymbol{C}$ matrix for testing the hypothesis $H_o : \mu_1 = \mu_2 = \mu_3$ vs $H_a : \mu_i \neq \mu_j$ for some $i \neq j$, where $\mu_i$ is the expectation of the $i$-th index.
   Answer: There are many ways to construct $\boldsymbol{C}$, a constrast matrix. For example,

$$\boldsymbol{C} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}.$$

   (c) Based on the $\boldsymbol{C}$ matrix of the prior question, compute $\boldsymbol{D} = \boldsymbol{CSC}'$.
   Answer: Based on the above choices of $\boldsymbol{C}$, we have

$$\boldsymbol{D} = \begin{bmatrix} 55 & 23 \\ 23 & 56 \end{bmatrix} \quad \text{or} \quad \boldsymbol{D} = \begin{bmatrix} 55 & -32 \\ -32 & 65 \end{bmatrix}$$

(d) The inverse of a $2 \times 2$ matrix $\boldsymbol{A}$ can be easily obtained as below:

$$\boldsymbol{A} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}, \Rightarrow \boldsymbol{A}^{-1} = \frac{1}{ac - b^2} \begin{bmatrix} c & -b \\ -b & a \end{bmatrix}.$$

Compute $\boldsymbol{D}^{-1}$.

Answer: $\boldsymbol{D}^{-1} = \begin{bmatrix} 0.0220 & -0.0090 \\ -0.0090 & 0.0216 \end{bmatrix}$ or $\boldsymbol{D}^{-1} = \begin{bmatrix} 0.0255 & 0.0125 \\ 0.0125 & 0.0216 \end{bmatrix}$.

(e) Perform the hypothesis $H_o : \boldsymbol{C\mu} = \boldsymbol{0}$ vs $H_a : \boldsymbol{C\mu} \neq \boldsymbol{0}$. You may use $F_{2,38}(0.95)$ $= 3.24$ to draw your conclusion.

Answer: For either choice of $\boldsymbol{C}$, we have $T^2 = 88.312$. The critical value is $\frac{(n-1)(q-1)}{n-q+1} F_{q-1,n-q+1}(0.95) = 6.65$. Therefore, reject the null hypothesis.

4. This problem is concerned with Chapter 7.

(a) Is $E(\hat{\boldsymbol{\beta}}) = \boldsymbol{\beta}$? Why?

Answer: Yes, because $E(\hat{\beta}) = E[(Z'Z)^{-1}(Z'\boldsymbol{Y})] = E[(Z'Z)^{-1}Z'(Z\boldsymbol{\beta} + \boldsymbol{\epsilon})] = \boldsymbol{\beta} + E[(Z'Z)^{-1}(Z'\boldsymbol{\epsilon})] = \boldsymbol{\beta}$.

(b) Let $e_j = y_j - \hat{y}_j$ be the least squares residual, where $\hat{y}_j = \hat{\beta}_0 + \hat{\beta}_1 z_{j1} + \hat{\beta}_2 z_{j2}$. Is $\sum_{j=1}^n e_j = 0$? Why?

Answer: Yes, because the model contains a constant term. As such, the system of least squares equation contains the equation

$$\sum_{j=1}^{n} (y_j - \beta_0 - \beta_1 z_{j1} - \beta_2 z_{j2}) = 0,$$

which is obtained by taking the partial derivative of the sum of squares with respect to $\beta_0$. This is basically $\sum_{j=1}^n (y_j - \hat{y}_j) = \sum_{j=1}^n e_j = 0$.

(c) (4 pts) Construct a least squares estimate of $\sigma^2$.

Answer: Based on the assumption, $\mathrm{Var}(\epsilon_j) = j\sigma^2$. Thus, a simple estimate of $\sigma^2$ is $\frac{1}{n-3} \sum_{j=1}^n \frac{e_j^2}{j}$. [You may use matrix notation to express the weights.]