1. (a) (6 points) $\bar{X} \sim N(\mu, \frac{1}{n}\Sigma)$

4.   (6 points) $Z_j \sim N(c'\mu, c'\Sigma c)$, where $c' = (-.5 \ -.5 \ 1 \ 0 \ 0)$. In this case

   $c'\mu = \mu_3 - (\mu_1 + \mu_2)$ and $c'\Sigma c = .25\sigma_{11} + .25\sigma_{22} + \sigma_{33} + .5\sigma_{12} - \sigma_{13} - \sigma_{23}$.

5. (8 points)   Since $n(\bar{X}-\mu)'S^{-1}(\bar{X}-\mu) = T^2 \sim \frac{p(n-1)}{n-p}F_{(p,n-p)}$, then

$$(\bar{X}-\mu)'S^{-1}(\bar{X}-\mu) \sim \frac{(5)(54)}{(55)(50)}F_{(5,50)}.$$

6. (10 points) Compute $t = \frac{r_{14\cdot 35}\sqrt{n-4}}{\sqrt{1-r_{14\cdot 35}^2}} = \frac{.21\sqrt{55-4}}{\sqrt{1-(.21)^2}} = 1.534$.

   Since $1.534 < t_{(54),.025} \cong 2$, we cannot reject $H_0 : \rho_{14\cdot 35} = 0$ at the 0.05 level
   of significance.

7. (10 points) Write the null hypothesis in matrix form. There is more than one way
   to do this correctly. One way to state the null hypothesis is

$$H_0 : C\mu = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{bmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

   Reject the null hypothesis that the difference between the post-course and pre-
   course means are the same for all three pre-course tests if

$$T^2 = 55(C\bar{X}-0)'(CSC')^{-1}(C\bar{X}-0) > \frac{(2)(54)}{53}F_{(2,53),\alpha}.$$

   A p-value is computed as $\Pr\{F_{(2,53)} > \frac{53}{(2)(53)}T^2\}$ where $F_{(2,53)}$ denotes a

   central F random variable with (2,53) degrees of freedom.

8. (10 points) Use the large sample chi-square approximation. Compute $-2\log(.18) = 3.43$. Do not reject the null hypothesis at the .05 level of significance because $3.43 < 9.49 = \chi_{4,.05}$. In this case we estimate $p + p(p+1)/2=20$ parameters under the alternative and $p+(p(p+1)/2)-4=16$ parameters under the null hypothesis, so the degrees of freedom for the large sample chi-square approximation to the distribution of this test statistics are 20-16=4.

2. (a) (6 points) The ANOVA table is

Source of Variation	df
Subjects	54
Differences	2
Error	108

The F-test has (2,108) degrees of freedom.

(b) (10 points)   Both tests assume

$$D_{\underset{\sim}{j}} = \begin{bmatrix} D_{1j} \\ D_{2j} \\ D_{3j} \end{bmatrix}$$ has a trivariate normal distribution and the subjects respond

independently from each other, so the $D_{\underset{\sim}{j}}$'s are independent   The F-test in part (a) of problem 2 is appropriate if $\mathrm{Var}(D_{ij} - D_{kj})$ is the same for all $i \neq k$. A special case of this is $\mathrm{Var}(D_{ij}) = \sigma_{\varepsilon}^2$ for all i=1,2,3, and the correlation between D_{ij} and D_{kj} is the same for all $i \neq k$. The T^2 test in part (e) of problem 1 does not place any restrictions on the variances of D_{1j}, D_{2j}, D_{3j} or the correlations among these three differences.

4. (a) (10 points) Use Bartlett's Test. Compute a polled estimate of the covariance matrix $S = \dfrac{(55-1)S_X + (22-1)S_Y}{(55-1)+(22-1)}$ with $(55-1)+(22-1) = 75$ degrees of freedom, $M = (75)\log(|S|) - (54)\log(|S_X|) - (21)\log(|S_Y|)$, and

$$C^{-1} = 1 - \left(\frac{2(5)^2 + (3)(5)-1}{6(5+1)(2-1)} \right)\left(\frac{1}{55-1} + \frac{1}{22-1} - \frac{1}{(55-1)+(22-1)} \right) = 0.9031922 \ .$$

Reject the null hypothesis of homogeneous covariance matrices at the α level of significance if $C^{-1}M > \chi_{(15),\,\alpha}$.

(b) (10 points) Do not use a pooled covariance matrix in this case. Reject the null hypothesis of equal population mean vectors if

$$T^2 = (\bar{X} - \bar{Y})' \left(\frac{1}{55} S_X + \frac{1}{22} S_Y \right)^{-1} (\bar{X} - \bar{Y}) > \frac{(5)(75)}{71} F_{(5,71),\alpha}.$$

This test statistic does not exactly have a the indicated multiple of an F-distribution when the null hypothesis is true, but the indicated rejection criterion keeps the Type I error level closer to α than the large sample chi-square approximation which would use $\chi^2_{(5),\alpha}$.

4. (14 points) Note the set of measurements on the proportion of time the j-th animal spends in each type of habitat $X_j = (X_{1j}\ X_{2j}\ X_{3j}\ X_{4j})'$ and the proportion of the j-th animal's home range that is covered by each type of habitat $Y_j = (Y_{1j}\ Y_{2j}\ Y_{3j}\ Y_{4j})'$ are repeated measurements on the same animal. One way to approach this problem is to make a joint 8-dimensional data vector for each animal in the study. The sample covariance matrix for these 8-dimensional data vectors would be singular, however, because the proportions are constrained to add to 1.0 in both $X_j = (X_{1j}\ X_{2j}\ X_{3j}\ X_{4j})'$ and $Y_j = (Y_{1j}\ Y_{2j}\ Y_{3j}\ Y_{4j})'$.

This problem can be eliminated by deleting the one proportion from $X_j = (X_{1j}\ X_{2j}\ X_{3j}\ X_{4j})'$ and deleting one proportion from $Y_j = (Y_{1j}\ Y_{2j}\ Y_{3j}\ Y_{4j})'$. It does not matter which proportion you delete, but here we will delete the last proportion from each data vector to obtain the combined data vector $Z_j = (X_{1j}\ X_{2j}\ X_{3j}\ Y_{1j}\ Y_{2j}\ Y_{3j})'$ for the j-th animal in the study. Compute $\bar{Z} = \sum_{j=1}^{30} Z_j$ and $S = \frac{1}{29} \sum_{j=1}^{30} (Z_j - \bar{Z})(Z_j - \bar{Z})'$. Then, the null hypothesis can be rejected if course tests if

$$T^2 = 30(C\bar{Z} - 0)'(CSC')^{-1}(C\bar{Z} - 0) > \frac{(3)(29)}{27} F_{(3,27),\alpha.},$$

where $C = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix}$.

A p-value is computed as $\Pr\{F_{(3,27)} > \dfrac{27}{(3)(29)} T^2\}$ where $F_{(3,27)}$ denotes a central F random variable with (3,27) degrees of freedom.

Alternatively, you could have used $\underset{\sim}{Z}_j = (X_{1j}\ X_{2j}\ X_{3j}\ X_{4j}\ Y_{1j}\ Y_{2j}\ Y_{3j}\ Y_{4j})'$ as the data vector for the j-th animal and performed the same T^2 test using

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}.$$

Alternatively, you could have used $\underset{\sim}{Z}_j = (X_{1j} - Y_{1j}\ \ X_{2j} - Y_{2j}\ \ X_{3j} - Y_{3j})'$ as the data vector for the j-th animal and performed the same T^2 test using

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

There are 100 possible points for your exam. There should be a score marked on your paper for each part of the exam. Also check if your total score was correctly on page 4 of your exam paper. Please return the exam to me if you think any errors were made in recording your score.

Exam Scores (100 possible points):

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10 | 0 0
 9 | 5 6
 9 | 0 1 2 4
 8 | 5 5 5 5 6 7 9
 8 | 0 0 4 4
 7 | 5 7 8 8 9
 7 | 0 2 4
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