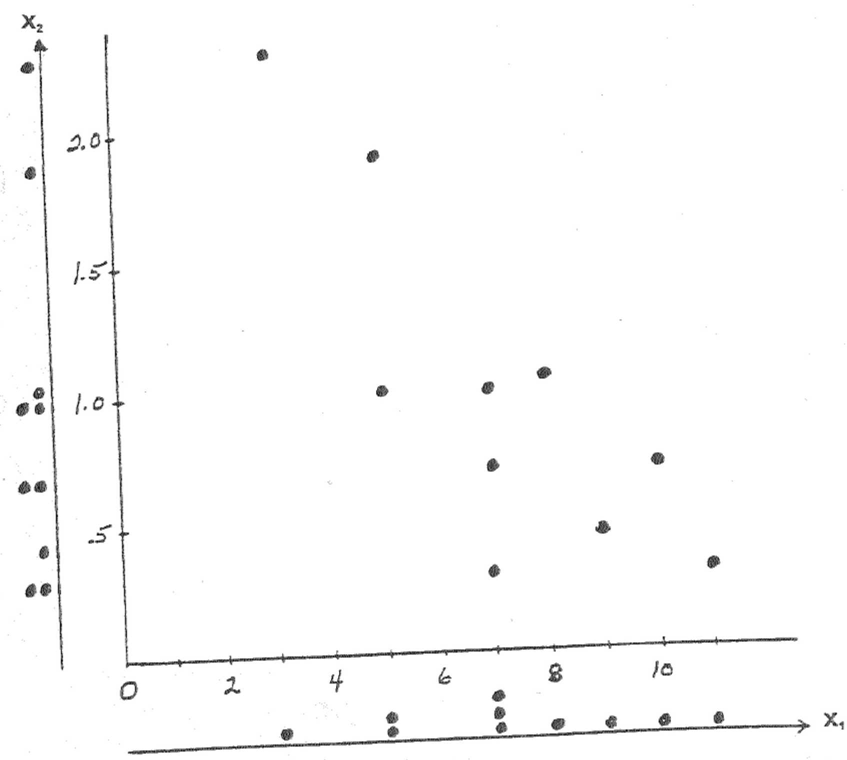
1.2 a)



b) s_{12} negative

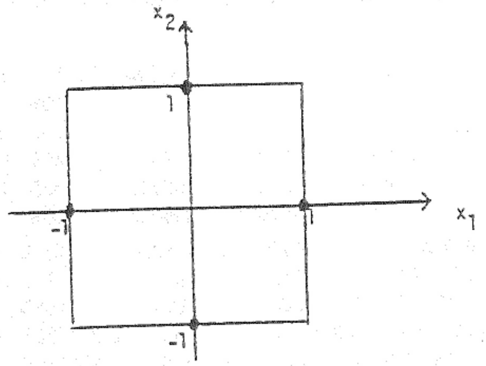
c) $\bar{x}_1 = 7.2$ $\qquad$ $\bar{x}_2 = .97$ $\qquad$ $s_{11} = 5.360$ $\qquad$ $s_{22} = .396$

$s_{12} = -1.169$ $\qquad$ $r_{12} = -.80$. Large x_1 occurs with small x_2 and vice versa.

d) $\bar{\underset{\sim}{x}} = \begin{bmatrix} 7.2 \\ .97 \end{bmatrix}$, $\quad S_n = \begin{bmatrix} 5.360 & -1.169 \\ -1.169 & .396 \end{bmatrix}$, $\quad R = \begin{bmatrix} 1 & -.80 \\ -.80 & 1 \end{bmatrix}$

1.12 a) If $P = (-3,4)$ then $d(0,P) = \max(|-3|,|4|) = 4$

b) The locus of points whose squared distance from (0,0) is 1 is



c) The generalization to p-dimensions is given by $d(0,P) = \max(|x_1|,|x_2|,\ldots,|x_p|)$

/ **1.17**

There are high positive correlations among all variables. The lowest correlation is 0.68557 between 200m and Marathon, and the highest correlation is 0.96917 bewteen 1500m and 3000m.

$$\bar{x} = \begin{pmatrix} 11.6185 \\ 23.6416 \\ 53.4058 \\ 2.0764 \\ 4.3255 \\ 9.4476 \\ 173.2533 \end{pmatrix}, \quad S_n \overset{\text{biased}}{=} \begin{pmatrix} 0.2008 & 0.4700 & 0.9926 & 0.0350 & 0.1075 & 0.2715 & 9.2726 \\ 0.4700 & 1.2120 & 2.5038 & 0.0855 & 0.2532 & 0.6383 & 22.7572 \\ 0.9926 & 2.5038 & 7.0431 & 0.2557 & 0.6887 & 1.6857 & 56.4471 \\ 0.0350 & 0.0855 & 0.2557 & 0.0115 & 0.0318 & 0.0756 & 2.5197 \\ 0.1075 & 0.2532 & 0.6887 & 0.0318 & 0.1085 & 0.2608 & 8.7193 \\ 0.2715 & 0.6383 & 1.6857 & 0.0756 & 0.2608 & 0.6672 & 22.1613 \\ 9.2726 & 22.7572 & 56.4471 & 2.5197 & 8.7193 & 22.1613 & 909.1216 \end{pmatrix},$$

$$R = \begin{pmatrix} 1.00000 & 0.95279 & 0.83469 & 0.72769 & 0.72837 & 0.74170 & 0.68634 \\ 0.95279 & 1.00000 & 0.85696 & 0.72406 & 0.69836 & 0.70987 & 0.68557 \\ 0.83469 & 0.85696 & 1.00000 & 0.89841 & 0.78784 & 0.77764 & 0.70542 \\ 0.72769 & 0.72406 & 0.89841 & 1.00000 & 0.90161 & 0.86357 & 0.77929 \\ 0.72837 & 0.69836 & 0.78784 & 0.90161 & 1.00000 & 0.96917 & 0.87793 \\ 0.74170 & 0.70987 & 0.77764 & 0.86357 & 0.96917 & 1.00000 & 0.89984 \\ 0.68634 & 0.68557 & 0.70542 & 0.77929 & 0.87793 & 0.89984 & 1.00000 \end{pmatrix}.$$

/ **2.6**

a) Since $A = A'$, A is symmetric.

b) Since the quadratic form

$$\underset{\sim}{x}'A\underset{\sim}{x} = [x_1, x_2]\begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 9x_1^2 - 4x_1x_2 + 6x_2^2$$

$$= (2x_1 - x_2)^2 + 5(x_1^2 + x_2^2) > 0 \quad \text{for} \quad [x_1, x_2] \neq [0,0]$$

we conclude that A is positive definite.

/ **2.7**

a) Eigenvalues: $\lambda_1 = 10$, $\lambda_2 = 5$.

Normalized eigenvectors: $\underset{\sim}{e}_1' = [2/\sqrt{5}, -1/\sqrt{5}] = [.894, -.447]$

$\underset{\sim}{e}_2' = [1/\sqrt{5}, 2/\sqrt{5}] = [.447, .894]$

b) $$A = \begin{bmatrix} 9 & -2 \\ -2 & \cancel{9}\,6 \end{bmatrix} = 10\begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}[2/\sqrt{5}, \ -1/\sqrt{5}] + 5\begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}[1/\sqrt{5}, \ 2/\sqrt{5}]$$

c) $$A^{-1} = \frac{1}{9(6)-(-2)(-2)}\begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix} = \begin{bmatrix} .12 & .04 \\ .04 & .18 \end{bmatrix}$$

d) Eigenvalues: $\lambda_1 = .2$, $\lambda_2 = .1$

Normalized eigenvectors: $\underset{\sim}{e}_1' = [1/\sqrt{5}, \ 2/\sqrt{5}]$

$\underset{\sim}{e}_2' = [2/\sqrt{5}, \ -1/\sqrt{5}]$

2.12 By (2-20), $A = P\Lambda P'$ with $PP' = P'P = I$. From Result 2A.11(e) $|A| = |P|\ |\Lambda|\ |P'| = |\Lambda|$. Since Λ is a diagonal matrix with diagonal elements $\lambda_1, \lambda_2, \ldots, \lambda_p$, we can apply Exercise 2.11 to get $|A| = |\Lambda| = \prod_{i=1}^{p} \lambda_i$.

2.13 Hint in the Text.

2.14 Let λ be an eigenvalue of A. Thus $0 = |A-\lambda I|$. If Q is orthogonal, $QQ' = I$ and $|Q||Q'| = 1$ by Exercise 2.13. Using Result 2A.11(e) we can then write

$$0 = |Q|\ |A-\lambda I|\ |Q'| = |QAQ'-\lambda I|$$

and it follows that λ is also an eigenvalue of QAQ' if Q is orthogonal.

2.16 $(A'A)' = A'(A')' = A'A$ showing $A'A$ is symmetric.

$$\underset{\sim}{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix} = A\underset{\sim}{x}. \quad \text{Then} \quad 0 \le y_1^2 + y_2^2 + \cdots + y_p^2 = \underset{\sim}{y}'\underset{\sim}{y} = \underset{\sim}{x}'A'A\underset{\sim}{x}$$

and $A'A$ is non-negative definite by definition.

2.17 10

suppose λ is the eigenvalue of a $k \times k$ positive definite matrix A. where

$$Ae = \lambda e$$

$$e'Ae = \lambda e'e$$

because A is positive definite matrix

$$\therefore e'Ae > 0$$

$$\Rightarrow \lambda e'e > 0$$

$$\lambda e^2 > 0 \quad (\because e^2 > 0)$$

$$\therefore \lambda > 0$$

$\Rightarrow$ every eigenvalue of a $k \times k$ positive definite matrix A is positive

2.18 Write $c^2 = \underset{\sim}{x}'A\underset{\sim}{x}$ with $A = \begin{bmatrix} 4 & -\sqrt{2} \\ \sqrt{2} & 3 \end{bmatrix}$. The eigenvalue-normalized eigenvector pairs for A are:

$$\lambda_1 = 2, \quad \underset{\sim}{e}_1' = [.577,\ \ .816]$$

$$\lambda_2 = 5, \quad \underset{\sim}{e}_2' = [.816,\ -.577]$$

For $c^2 = 1$, the half lengths of the major and minor axes of the ellipse of constant distance are

$$\frac{c}{\sqrt{\lambda_1}} = \frac{1}{\sqrt{2}} = .707 \quad \text{and} \quad \frac{c}{\sqrt{\lambda_2}} = \frac{1}{\sqrt{5}} = .447$$

respectively. These axes lie in the directions of the vectors $\underset{\sim}{e}_1$ and $\underset{\sim}{e}_2$ respectively.

For $c^2 = 4$, the half lengths of the major and minor axes are

$$\frac{c}{\sqrt{\lambda_1}} = \frac{2}{\sqrt{2}} = 1.414 \quad \text{and} \quad \frac{c}{\sqrt{\lambda_2}} = \frac{2}{\sqrt{5}} = .894\ .$$

As c^2 increases the lengths of the major and minor axes increase.

2.25

a) $V^{1/2} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$; $\underset{\sim}{\rho} = \begin{bmatrix} 1 & -1/5 & 4/15 \\ -1/5 & 1 & 1/6 \\ 4/15 & 1/6 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -.2 & .267 \\ -.2 & 1 & .167 \\ .267 & .167 & 1 \end{bmatrix}$

b) $V^{1/2} \underset{\sim}{\rho} V^{1/2} =$

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1/5 & 4/15 \\ -1/5 & 1 & 1/6 \\ 4/15 & 1/6 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 5 & -1 & 4/3 \\ -2/5 & 2 & 1/3 \\ 4/5 & 1/2 & 3 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 25 & -2 & 4 \\ -2 & 4 & 1 \\ 4 & 1 & 9 \end{bmatrix} = \underset{\sim}{\Sigma}$$

2.27

a) $\mu_1 - 2\mu_2$, $\sigma_{11} + 4\sigma_{22} - 4\sigma_{12}$

b) $-\mu_1 + 3\mu_2$, $\sigma_{11} + 9\sigma_{22} - 6\sigma_{12}$

c) $\mu_1 + \mu_2 + \mu_3$, $\sigma_{11} + \sigma_{22} + \sigma_{33} + 2\sigma_{12} + 2\sigma_{13} + 2\sigma_{23}$

d) $\mu_1 + 2\mu_2 - \mu_3$, $\sigma_{11} + 4\sigma_{22} + \sigma_{33} + 4\sigma_{12} - 2\sigma_{13} - 4\sigma_{23}$

e) $3\mu_1 - 4\mu_2$, $9\sigma_{11} + 16\sigma_{22}$ since $\sigma_{12} = 0$.

2.30

(a) $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$

(b) 10

(c) $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$

(d) 7

(e) $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

(f) $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$

(g) $\begin{pmatrix} 9 & -2 \\ -2 & 4 \end{pmatrix}$

(h) $\begin{pmatrix} 33 & 36 \\ 36 & 48 \end{pmatrix}$

(i) $\begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix}$

(j) $(0, 6)$

2.34 $\mathbf{b}'\mathbf{b} = 4 + 1 + 16 + 0 = 21$, $\mathbf{d}'\mathbf{d} = 15$ and $\mathbf{b}'\mathbf{d} = -2-3-8+0 = -13$

$$(\mathbf{b}'\mathbf{d})^2 = 169 \le 21(15) = 315$$

2.35 $\mathbf{b}'\mathbf{d} = -4+3 = -1$

$$\mathbf{b}'B\mathbf{b} = [-4, 3]\begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} -4 \\ 3 \end{bmatrix} = [-14 \quad 23]\begin{bmatrix} -4 \\ 3 \end{bmatrix} = 125$$

$$\mathbf{d}'B^{-1}\mathbf{d} = [1,1]\begin{bmatrix} 5/6 & 2/6 \\ 2/6 & 2/6 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 11/6$$

so $1 = (\mathbf{b}'\mathbf{d})^2 \le 125\ (11/6) = 229.17$

2.36

$$4x_1^2 + 4x_2^2 + 6x_1x_2 = \mathbf{x}'A\mathbf{x} \text{ where } A = \begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}.$$

$(4-\lambda)^2 - 3^2 = 0$ gives $\lambda_1 = 7, \lambda_2 = 1$. Hence the maximum is 7 and the minimum is 1.

2.37 From (2-51), $\max\limits_{\mathbf{x}'\mathbf{x}=1} \mathbf{x}'A\mathbf{x} = \max\limits_{\mathbf{x}\ne\mathbf{0}} \dfrac{\mathbf{x}'A\mathbf{x}}{\mathbf{x}'\mathbf{x}} = \lambda_1$

where λ_1 is the largest eigenvalue of A. For A given in Exercise 2.6, we have from Exercise 2.7, $\lambda_1 = 10$ and $\mathbf{e}_1' = [.894, -.447]$. Therefore $\max\limits_{\mathbf{x}'\mathbf{x}=1} \mathbf{x}'A\mathbf{x} = 10$ and this maximum is attained for $\mathbf{x} = \mathbf{e}_1$.

2.39 Hint in the Text.

2.40 Hint in the Text.