3.1

a) $\underset{\sim}{\bar{x}} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

b) $\underset{\sim}{e}_1 = \underset{\sim}{y}_1 - \bar{x}_1 \underset{\sim}{1} = [4, 0, -4]'$

$\underset{\sim}{e}_2 = \underset{\sim}{y}_2 - \bar{x}_2 \underset{\sim}{1} = [-1, 1, 0]'$

c) $L_{\underset{\sim}{e}_1} = \sqrt{32}$; $L_{\underset{\sim}{e}_2} = \sqrt{2}$

Let θ be the angle between $\underset{\sim}{e}_1$ and $\underset{\sim}{e}_2$, then $\cos(\theta) = -4/\sqrt{32 \times 2} = -.5$

Therefore $n\, s_{11} = L^2_{\underset{\sim}{e}_1}$ or $s_{11} = 32/3$; $n\, s_{22} = L^2_{\underset{\sim}{e}_2}$ or $s_{22} = 2/3$; $n\, s_{12} = \underset{\sim}{e}_1' \underset{\sim}{e}_2$ or $s_{12} = -4/3$. Also, $r_{12} = \cos(\theta) = -.5$. Consequently $S_n = \begin{bmatrix} 32/3 & -4/3 \\ -4/3 & 2/3 \end{bmatrix}$ and $R = \begin{bmatrix} 1 & -.5 \\ -.5 & 1 \end{bmatrix}$.

3.7 All ellipses are centered at $\underset{\sim}{\bar{x}}$.

i) For $S = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$, $S^{-1} = \begin{bmatrix} 5/9 & -4/9 \\ -4/9 & 5/9 \end{bmatrix}$

Eigenvalue-normalized eigenvector pairs for S^{-1} are:

$\lambda_1 = 1$, $\underset{\sim}{e}_1' = [.707, \; -.707]$

$\lambda_2 = 1/9$, $\underset{\sim}{e}_2' = [.707, \; .707]$

Half lengths of axes of ellipse $(\underset{\sim}{x} - \underset{\sim}{\bar{x}})' S^{-1} (\underset{\sim}{x} - \underset{\sim}{\bar{x}}) \le 1$ are $1/\sqrt{\lambda_1} = 1$ and $1/\sqrt{\lambda_2} = 3$ respectively. The major axis of ellipse lies in the direction of $\underset{\sim}{e}_2$; the minor axis lies in the direction of $\underset{\sim}{e}_1$.

ii) For $S = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}$, $S^{-1} = \begin{bmatrix} 5/9 & 4/9 \\ 4/9 & 5/9 \end{bmatrix}$

Eigenvalue-normalized eigenvectors for S^{-1} are:

$\lambda_1 = 1$, $\underset{\sim}{e}_1' = [.707, \; .707]$

$\lambda_2 = 1/9$, $\underset{\sim}{e}_2' = [.707, \; -.707]$

3.9 (a) We calculate $\overline{x} = [16, 18, 34]'$ and

$$X_c = \begin{bmatrix} -4 & -1 & -5 \\ 2 & 2 & 4 \\ -2 & -2 & -4 \\ 4 & 0 & 4 \\ 0 & 1 & 1 \end{bmatrix}$$ and we notice $\text{col}_1(X_c) + \text{col}_2(X_c) = \text{col}_1(X_c)$

so $a = [1, 1, -1]'$ gives $X_c a = 0$.

(b)

$$S = \begin{bmatrix} 10 & 3 & 13 \\ 3 & 2.5 & 5.5 \\ 13 & 5.5 & 18.5 \end{bmatrix} \quad \text{so } |S| = \begin{matrix} 10(2.5)(18.5) & + & 39(15.5) & + & 39(15.5) \\ -(13)^2(2.5) & - & 9(18.5) & - & 55(5.5) \end{matrix} = 0$$

As above in a)

$$Sa = \begin{bmatrix} 10 + 3 - 13 \\ 3 + 2.5 - 5.5 \\ 13 + 5.5 - 18.5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(c) Check.

3.14 a) From first principles we have

$$\underset{\sim}{b}' \underset{\sim}{x}_1 = [2 \quad 3] \begin{bmatrix} 9 \\ 1 \end{bmatrix} = 21$$

Similarly $\underset{\sim}{b}' \underset{\sim}{x}_2 = 19$ and $\underset{\sim}{b}' \underset{\sim}{x}_3 = 8$ so

$$\text{sample mean} = \frac{21+19+8}{3} = 16$$

$$\text{sample variance} = \frac{(21-16)^2+(19-16)^2+(8-16)^2}{2} = 49$$

Also $\underset{\sim}{c}' \underset{\sim}{x}_1 = [-1 \quad 2] \begin{bmatrix} 9 \\ 1 \end{bmatrix} = -7$; $\underset{\sim}{c}' \underset{\sim}{x}_2 = 1$ and $\underset{\sim}{c}' \underset{\sim}{x}_3 = 3$

so

sample mean = -1

sample variance = 28

Finally sample covariance $= \dfrac{(21-16)(-7+1)+(19-16)(1+1)+(8-16)(3+1)}{2} =$
-28.

b) $\bar{\underset{\sim}{x}}' = [5 \quad 2]$ and $S = \begin{bmatrix} 16 & -2 \\ -2 & 1 \end{bmatrix}$

Using (3-36)

3.16 Since $\Sigma_V = E(\underset{\sim}{V} - \underset{\sim}{\mu}_V)(\underset{\sim}{V} - \underset{\sim}{\mu}_V)'$

$$= E(\underset{\sim}{V}\underset{\sim}{V}' - \underset{\sim}{V}\underset{\sim}{\mu}_V' - \underset{\sim}{\mu}_V\underset{\sim}{V}' + \underset{\sim}{\mu}_V\underset{\sim}{\mu}_V')$$

$$= E(\underset{\sim}{V}\underset{\sim}{V}') - E(\underset{\sim}{V})\underset{\sim}{\mu}_V' - \underset{\sim}{\mu}_V E(\underset{\sim}{V}') + \underset{\sim}{\mu}_V\underset{\sim}{\mu}_V'$$

$$= E(\underset{\sim}{V}\underset{\sim}{V}') - \underset{\sim}{\mu}_V\underset{\sim}{\mu}_V' - \underset{\sim}{\mu}_V\underset{\sim}{\mu}_V' + \underset{\sim}{\mu}_V\underset{\sim}{\mu}_V'$$

$$= E(\underset{\sim}{V}\underset{\sim}{V}') - \underset{\sim}{\mu}_V\underset{\sim}{\mu}_V',$$

we have $E(\underset{\sim}{V}\underset{\sim}{V}') = \Sigma + \underset{\sim}{\mu}_V\underset{\sim}{\mu}_V'$.

3.17 Hint in the Text.

4.2 (a) We are given $p = 2$, $\mu = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, $\Sigma = \begin{bmatrix} 2 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 1 \end{bmatrix}$ so $|\Sigma| = 3/2$

and

$$\Sigma^{-1} = \begin{bmatrix} \frac{2}{3} & -\frac{\sqrt{2}}{3} \\ -\frac{\sqrt{2}}{3} & \frac{4}{3} \end{bmatrix}$$

$$f(x) = \frac{1}{(2\pi)\sqrt{3/2}} \exp\left(-\frac{1}{2} \left[\frac{2}{3}x_1^2 - \frac{2\sqrt{2}}{3}x_1(x_2 - 2) + \frac{4}{3}(x_2 - 2)^2 \right] \right)$$

(b)

$$\frac{2}{3}x_1^2 - \frac{2\sqrt{2}}{3}x_1(x_2 - 2) + \frac{4}{3}(x_2 - 2)^2$$

(c) $c^2 = \chi_2^2(.5) = 1.39$. Ellipse centered at $[0, 2]'$ with the major axis having half-length $\sqrt{\lambda_1}\ c = \sqrt{2.366}\sqrt{1.39} = 1.81$. The major axis lies in the direction $e = [.888, .460]'$. The minor axis lies in the direction $e = [-.460, .888]'$ and has half-length $\sqrt{\lambda_2}\ c = \sqrt{.634}\sqrt{1.39} = .94$.

4.3 We apply Result 4.5 that relates zero covariance to statistical independence

a) No, $\sigma_{12} \neq 0$

b) Yes, $\sigma_{23} = 0$

c) Yes, $\sigma_{13} = \sigma_{23} = 0$

d) Yes, by Result 4.3, $(X_1+X_2)/2$ and X_3 are jointly normal and their covariance is $\frac{1}{2}\sigma_{13} + \frac{1}{2}\sigma_{23} = 0$.

e) No, by Result 4.3 with $A = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{5}{2} & 1 & -1 \end{bmatrix}$, form $A \Sigma A'$ to see that the covariance is 10 and not 0.

$$\Sigma' = \begin{bmatrix} 5 & 10 \\ 10 & \frac{93}{4} \end{bmatrix}$$

4.4 a) $3X_1 - 2X_2 + X_3$ is $N(13,9)$

b) Require Cov $(X_2, X_2 - a_1X_1 - a_3X_3) = 3 - a_1 - 2a_3 = 0$. Thus any $\underset{\sim}{a}' = [a_1, a_3]$ of the form $\underset{\sim}{a}' = [3-2a_3, a_3]$ will meet the requirement. As an example, $\underset{\sim}{a}' = [1,1]$.

4.5 a) $X_1|x_2$ is $N(\frac{1}{\sqrt{2}}(x_2-2), \frac{3}{2})$

b) $X_2|x_1, x_3$ is $N(-2x_1-5, 1)$

c) $X_3|x_1, x_2$ is $N(\frac{1}{2}(x_1+x_2+3), \frac{1}{2})$

4.10 Hint in the Text.

4.11 Hint in the Text.

4.13

(a) By using the exercise 4.11, we can write that

$$|\Sigma| = |\Sigma_{22}||\Sigma_11 - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}| \qquad \text{for } |\Sigma_{22} \neq 0|$$

(b)

$$\Sigma^{-1} = \begin{bmatrix} I & 0 \\ -\Sigma_{22}^{-1}\Sigma_{21} & I \end{bmatrix} \begin{bmatrix} (\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})^{-1} & 0 \\ 0' & \Sigma_{22}^{-1} \end{bmatrix} \begin{bmatrix} I & -\Sigma_{12}\Sigma_{22}^{-1} \\ 0 & I \end{bmatrix}$$

$$\begin{aligned}
(x-\mu)'\Sigma^{-1}(x-\mu) &= \begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{bmatrix} \begin{bmatrix} I & 0 \\ -\Sigma_{22}^{-1}\Sigma_{21} & I \end{bmatrix} \\
&\times \begin{bmatrix} (\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})^{-1} & 0 \\ 0' & \Sigma_{22}^{-1} \end{bmatrix} \begin{bmatrix} I & -\Sigma_{12}\Sigma_{22}^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{bmatrix} \\
&= \begin{bmatrix} x_1 - \mu_1 - \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2) \\ x_2 - \mu_2 \end{bmatrix}' \begin{bmatrix} (\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})^{-1} & 0 \\ 0' & \Sigma_{22}^{-1} \end{bmatrix} \\
&\times \begin{bmatrix} x_1 - \mu_1 - \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2) \\ x_2 - \mu_2 \end{bmatrix} \\
&= [x_1 - \mu_1 - \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2)]'(\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})^{-1} \\
&\times [x_1 - \mu_1 - \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2)] + (x_2 - \mu_2)'\Sigma_{22}^{-1}(x_2 - \mu_2)
\end{aligned}$$

(c) The marginal distribution of X_2

$$\begin{aligned}
&(x_2 - \mu_2)'\Sigma_{22}^{-1}(x_2 - \mu_2) \\
&\Rightarrow N(\mu_2, \Sigma_{22})
\end{aligned}$$

The conditional distribution of $X_1|X_2 = x_2$

$$\begin{aligned}
&[x_1 - \mu_1 - \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2)]'(\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})^{-1}[x_1 - \mu_1 - \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2)] \\
&\Rightarrow N(\mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2), \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})
\end{aligned}$$

4.18 By Result 4.11 we know that the maximum likelihood estimates of $\underset{\sim}{\mu}$ and $\underset{\sim}{\Sigma}$ are $\bar{\underset{\sim}{x}} = [4,6]'$ and

$$\frac{1}{n}\sum_{j=1}^{n}(\underset{\sim}{x}_j-\bar{\underset{\sim}{x}})(\underset{\sim}{x}_j-\bar{\underset{\sim}{x}})' = \frac{1}{4}\left\{\left(\begin{bmatrix}3\\6\end{bmatrix}-\begin{bmatrix}4\\6\end{bmatrix}\right)\left(\begin{bmatrix}3\\6\end{bmatrix}-\begin{bmatrix}4\\6\end{bmatrix}\right)'+\left(\begin{bmatrix}4\\4\end{bmatrix}-\begin{bmatrix}4\\6\end{bmatrix}\right)\left(\begin{bmatrix}4\\4\end{bmatrix}-\begin{bmatrix}4\\6\end{bmatrix}\right)'\right.$$

$$\left.+\left(\begin{bmatrix}5\\7\end{bmatrix}-\begin{bmatrix}4\\6\end{bmatrix}\right)\left(\begin{bmatrix}5\\7\end{bmatrix}-\begin{bmatrix}4\\6\end{bmatrix}\right)'+\left(\begin{bmatrix}4\\7\end{bmatrix}-\begin{bmatrix}4\\6\end{bmatrix}\right)\left(\begin{bmatrix}4\\7\end{bmatrix}-\begin{bmatrix}4\\6\end{bmatrix}\right)'\right\}$$

$$= \frac{1}{4}\left\{\begin{bmatrix}-1\\0\end{bmatrix}\begin{bmatrix}-1 & 0\end{bmatrix}+\begin{bmatrix}0\\-2\end{bmatrix}\begin{bmatrix}0 & -2\end{bmatrix}+\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix}\right\}$$

$$= \frac{1}{4}\begin{bmatrix}2 & 1\\1 & 6\end{bmatrix}$$

4.21 (a) $\overline{X}$ is distributed $N_4(\mu, n^{-1}\Sigma)$

(b) $X_1-\mu$ is distributed $N_4(0,\Sigma)$ so $(X_1-\mu)'\Sigma^{-1}(X_1-\mu)$ is distributed as chi-square with p degrees of freedom.

(c) Using Part a),

$$(\overline{X}-\mu)'(n^{-1}\Sigma)^{-1}(\overline{X}-\mu) = n(\overline{X}-\mu)'\Sigma^{-1}(\overline{X}-\mu)$$

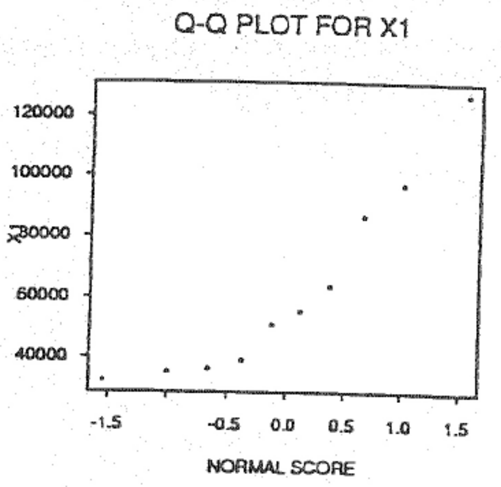
is distributed as chi-square with p degrees of freedom.

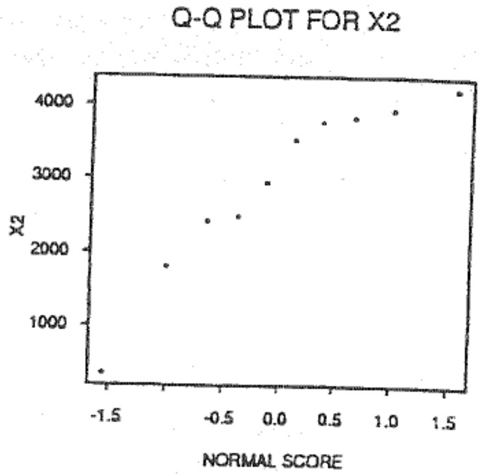
(d) Approximately distributed as chi-square with p degrees of freedom. Since the sample size is large, Σ can be replaced by S.

4.24

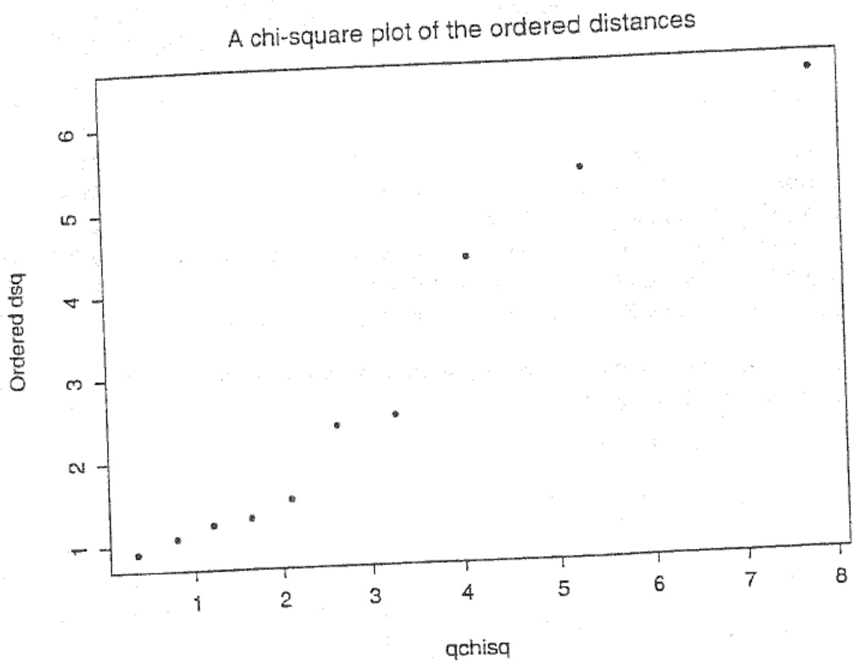
(a). Q-Q plots are given below. X_1 has long left hand tails, but X_2 has a long right hand tail.

(b). The critical point for $\alpha = 0.10$ when $n = 10$ is 0.9351. Calculated values of r_Q for X_1, X_2 are 0.93617, 0.94798 respectively. Since they are larger than 0.9351, we do not reject the hypothesis of normality for each variable.





4.25. 10 largest U. S. industrial corporations data



XBAR

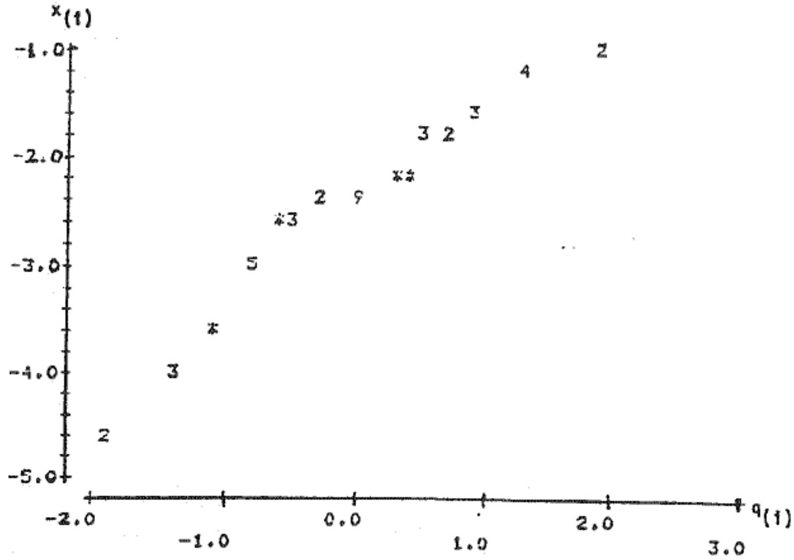
62309.1
2927.3
81248.4

S

	sales	profits	assets
	1000509114	25575600	1511827230
	25575600	1430020	45654618
	1511827230	45654618	2980489782

	Ordered dsq	qchisq
1	0.9078	0.3518
2	1.0749	0.7978
3	1.2284	1.2125
4	1.2963	1.6416
5	1.5003	2.1095
6	2.3614	2.6430
7	2.4713	3.2831
8	4.3524	4.1083
9	5.3677	5.3170
10	6.4396	7.8147

4.27 Q-Q plot is shown below.



The Q-Q plot is reasonably straight. $r_Q = .978$ (λ=0)

For λ = 1/4, $r_Q = .993$ so λ = 1/4 is a little better choice for the normalizing transformation.

4.30

(a). $\hat{\lambda}_1 = 1.0$. There is no need for transformation. $r_Q = 0.98659 > 0.9198$ where 0.9198 is the critical point for testing normality with $\alpha = 0.05$ when n=10. We accept the null hypothesis of normality.

(b). $\hat{\lambda}_2 = 0.5$. After the transformation, $r_Q = 0.97546 > 0.9198$ where 0.9198 is the critical point for testing normality with $\alpha = 0.05$ when n=10. We accept the null hypothesis of normality.

(c). $(\hat{\lambda}_1, \hat{\lambda}_2) = (0.5, 0.4)$. They are different from results obtained in (a), (b) because the likelihood is flat due to the small sample size.

4.36

Marginal Normality:

	X_1	X_2	X_3	X_4	X_5	X_6	X_7
r_Q	0.98674	0.98496	0.99561	0.98554	0.92155*	0.92558*	0.87316*

*: significant at 5 % level (the critical point = 0.9787 for n=55).

Multivariate Normality: the χ^2 plot does not look like normal.

PROBLEM 4.36