STAT 501 **Assignment 1** **Name ________________**
Spring 2005

Reading Assignment:

Johnson and Wichern, Chapter 1, Sections 2.5 and 2.6, Chapter 4, and Chapter 3. Review matrix operations in Chapter 2 and Supplement 2A.

Written Assignment: Due Monday, January 24, in class.

1. Consider a random vector $\underset{\sim}{X} = (X_1\ X_2\ X_3)'$ with mean vector $\underset{\sim}{\mu} = (3,\ 2,\ 1)'$ and covariance matrix Σ. The eigenvalues of Σ are $\lambda_1 = 12$, $\lambda_2 = 6$, $\lambda_3 = 2$ and the corresponding eigenvectors are

$$\underset{\sim}{e}_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \qquad \underset{\sim}{e}_2 = \begin{bmatrix} 2/\sqrt{6} \\ -1/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix} \qquad \underset{\sim}{e}_3 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

Evaluate the following quantities. Parts (a), (b), (c), (d) and (g) should be done without using a computer.

(a) $|\Sigma| =$ _____ (b) trace(Σ) = _____ (c) $V(\underset{\sim}{e}'_1 \underset{\sim}{X}) =$ _____

(d) $\Sigma = \begin{bmatrix} \quad \end{bmatrix}$ (e) $\Sigma^{-1} = \begin{bmatrix} \quad \end{bmatrix}$

(f) $\Sigma^{-1/2} = \begin{bmatrix} \quad \end{bmatrix}$

(g) Define $Y_1 = \underset{\sim}{e}_1'\underset{\sim}{X}$, $Y_2 = \underset{\sim}{e}_2'\underset{\sim}{X}$, and $Y_3 = \underset{\sim}{e}_3'\underset{\sim}{X}$. Find the mean vector and covariance matrix for $\underset{\sim}{Y} = (Y_1\ Y_2\ Y_3)' = \Gamma'\underset{\sim}{X}$, where the i-th column of Γ is the i-th eigenvector for Σ. Evaluate

$$E(\underset{\sim}{Y}) = \begin{bmatrix} \\ \\ \\ \end{bmatrix} \qquad V(\underset{\sim}{Y}) = \Gamma'\Sigma\Gamma = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

(h) The linear transformation, $\underset{\sim}{Y} = \Gamma'\underset{\sim}{X}$, examined in part (g) is often called a rotation because it corresponds to simply rotating the coordinate axes. Use the definition of eigenvectors to show that lengths of vectors are not changed by this transformation, i.e., show $\underset{\sim}{Y}'\underset{\sim}{Y} = \underset{\sim}{X}'\underset{\sim}{X}$ when $\underset{\sim}{Y} = \Gamma'\underset{\sim}{X}$.

(i) Let Γ be a matrix for which the i-th column of Γ is the i-th eigenvector for Σ, and let $\underset{\sim}{Y} = \Gamma'\underset{\sim}{X}$, $\underset{\sim}{\mu}_X = E(\underset{\sim}{X})$, $\Sigma_X = V(\underset{\sim}{X})$, $\underset{\sim}{\mu}_Y = \Gamma'\underset{\sim}{\mu}_X$, and $\Sigma_Y = V(\underset{\sim}{Y}) = \Gamma'\Sigma_X\Gamma$. Determine which of the following measures of variability or distance are unaffected by rotations. Justify your answers by either giving a counter example or a proof using the properties of eigenvalues, eigenvectors, and matrix operations given in chapter 2 of Johnson and Wichern,

(a) Is $|\Sigma_X| = |\Sigma_Y|$?

(b) Is trace (Σ_X) = trace (Σ_Y) ?

(c) Is $\Sigma_X = \Sigma_Y$?

(d) Is $(\underset{\sim}{X} - \underset{\sim}{\mu}_X)'\Sigma_X^{-1}(\underset{\sim}{X} - \underset{\sim}{\mu}_X) = (\underset{\sim}{Y} - \underset{\sim}{\mu}_Y)'\Sigma_Y^{-1}(\underset{\sim}{Y} - \underset{\sim}{\mu}_Y)$?

(e) Is $(\underset{\sim}{X} - \underset{\sim}{\mu}_X)'(\underset{\sim}{X} - \underset{\sim}{\mu}_X) = (\underset{\sim}{Y} - \underset{\sim}{\mu}_Y)'(\underset{\sim}{Y} - \underset{\sim}{\mu}_Y)$?

2. Consider a bivariate normal population with mean vector $\underset{\sim}{\mu}$ and covariance matrix Σ, where

$$\underset{\sim}{\mu} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{and} \quad \Sigma = \begin{bmatrix} 5 & -3 \\ -3 & 9 \end{bmatrix}$$

(a) Write down a formula for the joint density function.

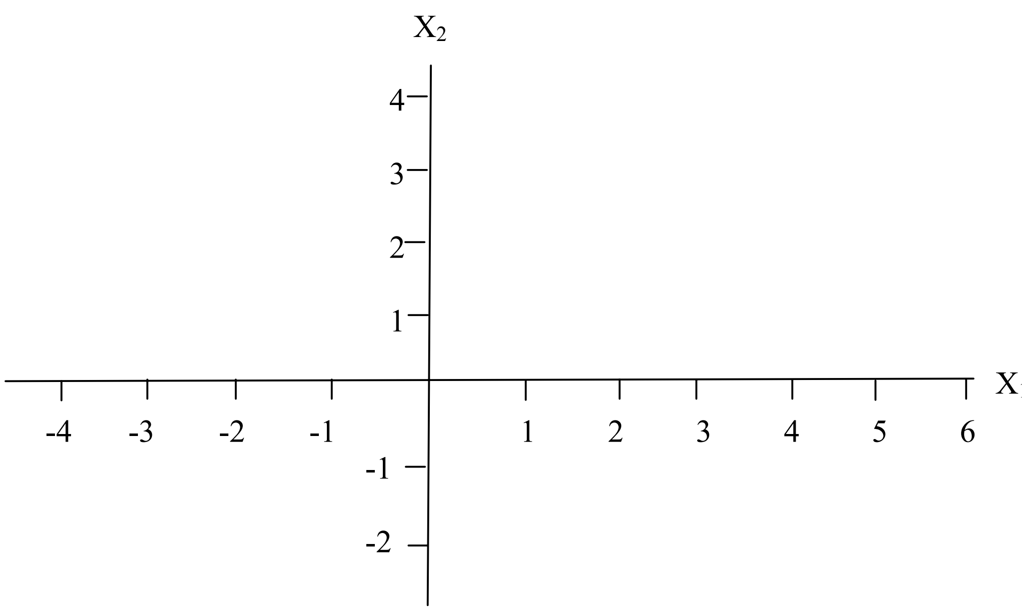
(b) Find the eigenvalues and eigenvectors for Σ.

λ_1 = ______ λ_2 = _____ $\underset{\sim}{e}_1 = \begin{pmatrix} \quad \end{pmatrix}$ $\underset{\sim}{e}_2 = \begin{pmatrix} \quad \end{pmatrix}$

(c) Find values for $|\Sigma|$ = _________ trace (Σ) = _________

(d) Write down the formula for the boundary of the smallest region such that there is probability 0.5 that a randomly selected observation will be inside the boundary.

(e) Sketch the boundary of the region you identified in part (d).



(f) Determine the area of the region described in parts (d) and (e). _______________

(g) Determine the area of the smallest region such that there is probability 0.95 that a randomly selected observation will be in that region. _______________

3. The joint density function for X_1 and X_2 is

$$f(x_1, x_2) = 3\pi^{-1} \exp\{-4.5x_1^2 + 3x_1x_2 - 2.5x_2^2 - 3x_1 + 17x_2 - 32.5\}$$

Find the mean vector and the covariance matrix for this bivariate distribution.

4. Let $\underset{\sim}{X}$ be a normally distributed random vector with

$$\underset{\sim}{\mu} = \begin{pmatrix} -3 \\ 1 \\ -2 \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} 4 & 0 & -1 \\ 0 & 5 & 0 \\ -1 & 0 & 2 \end{pmatrix}$$

(a) Which of the following are pairs of independent random variables?

(i) X_1 and X_2

(ii) X_1 and X_3

(iii) X_1 and $X_1 + 3X_2 - 2X_3$

(iv) (X_1, X_3) and X_2

(v) $X_1 + X_3$ and $X_1 - 2X_2$

(vi) $X_1 + X_2 + X_3$ and $4X_1 - 2X_2 + 3X_3$

(b) What is the distribution of $\underset{\sim}{Y} = (X_1, X_2)'$?

(c) What is the conditional distribution of $\underset{\sim}{Y} = (X_1, X_2)'$ given $X_3 = x_3$?

(d) Find the correlation between X_1 and X_2 and a formula for the partial correlation between X_1 and X_2 given $X_3 = x_3$.

$\rho_{12} =$ ____________ $\rho_{12\bullet 3} =$ _______________

(e) What is the conditional distribution of X_2 given $X_1 = x_1$ and $X_3 = x_3$?

5. Suppose $\underset{\sim}{X} \sim N_4\left(\underset{\sim}{\mu}, \underset{\sim}{\Sigma}\right)$ where

$$\underset{\sim}{\mu} = \begin{pmatrix} -4 \\ 3 \\ 0 \\ 1 \end{pmatrix} \qquad \Sigma = \begin{pmatrix} 4 & 0 & 2 & 2 \\ 0 & 1 & 1 & 0 \\ 2 & 1 & 9 & -2 \\ 2 & 0 & -2 & 4 \end{pmatrix}$$

(a) What is the distribution of $Z_1 = 3X_1 - 5X_2 + X_4$?

(b) What is the joint distribution of Z_1 in part (a) and $Z_2 = 2X_1 - X_3 + 3X_4$.

(c) Find the conditional distribution of $\begin{pmatrix} X_1 \\ X_4 \end{pmatrix}$ given $X_3 = 1$.

(d) Find the partial correlation between X_1 and X_4 given $X_2 = x_2$ and $X_3 = x_3$.

6. Let $\underset{\sim}{X}$ be $N\left(\underset{\sim}{\mu_1}, \Sigma_1\right)$ and $\underset{\sim}{Y}$ be $N\left(\underset{\sim}{\mu_2}, \Sigma_2\right)$ where $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ are independent and

$$\underset{\sim}{\mu_1} = \begin{pmatrix} 2 \\ 6 \end{pmatrix} \qquad \Sigma_1 = \begin{pmatrix} 8 & -2 \\ -2 & 4 \end{pmatrix} \qquad \underset{\sim}{\mu_2} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \qquad \Sigma_2 = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$$

(a) Evaluate $\text{Cov}\left(\underset{\sim}{X} - \underset{\sim}{Y},\ \underset{\sim}{X} + \underset{\sim}{Y}\right)$

(b) Are $\underset{\sim}{X} - \underset{\sim}{Y}$ and $\underset{\sim}{X} + \underset{\sim}{Y}$ independent random vectors? Explain.

(c) Show that the joint distribution for the four dimensional random vector

$\begin{bmatrix} \underset{\sim}{X} - \underset{\sim}{Y} \\ \underset{\sim}{X} + \underset{\sim}{Y} \end{bmatrix}$ is a multivariate normal distribution.

7. Let $d(\underset{\sim}{p}, \underset{\sim}{q})$ denote a measure of distance between $\underset{\sim}{p}$ and $\underset{\sim}{q}$. Johnson and Wichern indicate that any distance measure should possess the following four properties:

(i) $d(\underset{\sim}{p}, \underset{\sim}{q}) = d(\underset{\sim}{q}, \underset{\sim}{p})$

(ii) $d(\underset{\sim}{p}, \underset{\sim}{q}) > 0$ if $\underset{\sim}{p} \neq \underset{\sim}{q}$

(iii) $d(\underset{\sim}{p}, \underset{\sim}{q}) = 0$ if $\underset{\sim}{p} = \underset{\sim}{q}$

(iv) $d(\underset{\sim}{p}, \underset{\sim}{q}) \leq d(\underset{\sim}{p}, \underset{\sim}{r}) + d(\underset{\sim}{r}, \underset{\sim}{q})$, where $\underset{\sim}{r} = (r_1, r_2)'$.

These properties are satisfied, for example, by Euclidean distance, i.e., .

$d(\underset{\sim}{p}, \underset{\sim}{q}) = [(\underset{\sim}{p} - \underset{\sim}{q})'(\underset{\sim}{p} - \underset{\sim}{q})]^{1/2}$

(a) In this class we will often consider distance measures of the form $d(\underset{\sim}{p}, \underset{\sim}{q}) = [(\underset{\sim}{p} - \underset{\sim}{q})' A (\underset{\sim}{p} - \underset{\sim}{q})]^{1/2}$. Sometimes A will be the inverse of a covariance matrix which implies that A is symmetric and positive definite. Show that properties (i) through (iv) are satisfied when A is symmetric and positive definite.

(b) Does A have to be both symmetric and positive definite for
$d(\underset{\sim}{p}, \underset{\sim}{q}) = [(\underset{\sim}{p} - \underset{\sim}{q})' A (\underset{\sim}{p} - \underset{\sim}{q})]^{1/2}$ to satisfy properties (i) through (iv)?

Present your proofs, counter examples or explanations.

(c) For k-dimensional vectors it is obvious that the measure

$$d(\underset{\sim}{p}, \underset{\sim}{q}) = \max\{|p_1 - q_1|, |p_2 - q_2|, \ldots, |p_k - q_k|\}$$

satisfies properties (i), (ii), (iii). Either prove or disprove that it satisfies property (iv), the triangle inequality.

8. (a) By multiplication of partitioned matrices, verify for yourself that

$$\begin{bmatrix} I_{q\times q} & B \\ 0_{r\times q} & I_{r\times r} \end{bmatrix}^{-1} = \begin{bmatrix} I_{q\times q} & -B \\ 0_{r\times q} & I_{r\times r} \end{bmatrix} \text{ and } \begin{bmatrix} I_{r\times r} & 0_{r\times q} \\ B & I_{q\times q} \end{bmatrix}^{-1} = \begin{bmatrix} I_{r\times r} & 0_{r\times q} \\ -B & I_{q\times q} \end{bmatrix}$$

for any $q \times r$ matrix B, where $0_{a\times b}$ denotes a matrix of zeros and $I_{a\times a}$ denotes an identity matrix.

(b) Show that $\begin{vmatrix} I_{q\times q} & B \\ 0_{r\times q} & I_{r\times r} \end{vmatrix} = \begin{vmatrix} I_{r\times r} & 0_{r\times q} \\ B & I_{q\times q} \end{vmatrix} = 1$ for any $q \times r$ matrix B.

(c) By multiplication of partitioned matrices verify that if $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ is a square matrix for which A_{11} is $q \times q$ matrix and A_{22} is an $r \times r$ matrix and the inverse of A_{22} exists, then

$$\begin{bmatrix} A_{11} - A_{12} A_{22}^{-1} A_{21} & 0_{q\times r} \\ 0_{r\times q} & A_{22} \end{bmatrix} = \begin{bmatrix} I_{q\times q} & -A_{12}A_{22}^{-1} \\ 0_{r\times q} & I_{r\times r} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} I_{q\times q} & 0_{q\times r} \\ -A_{22}^{-1} A_{21} & I_{r\times r} \end{bmatrix}$$

(d) Use the results from parts (a) through (c) to show that

$|A| = |A_{22}| \left|A_{11} - A_{12} A_{22}^{-1} A_{21}\right|$ for any square matrix A with $|A_{22}| \neq 0$.

(e) Use the results from parts (a) through (d) to show that

$$A^{-1} = \begin{bmatrix} I_{q\times q} & 0_{q\times r} \\ -A_{22}^{-1}A_{21} & I_{r\times r} \end{bmatrix}^{-1} \begin{bmatrix} \left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)^{-1} & 0_{q\times r} \\ 0_{r\times q} & A_{22}^{-1} \end{bmatrix} \begin{bmatrix} I_{q\times q} & -A_{12}A_{22}^{-1} \\ 0_{r\times q} & I_{r\times r} \end{bmatrix}^{-1}$$

for any symmetric matrix A with $|A_{22}| \neq 0$.

9. Consider a multivariate normal distribution with positive definite covariance matrix $\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$. Note that from part (d) in problem 8, $|\Sigma| = |\Sigma_{22}|\left|\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\right|$, the product of determinants of covariance matrices for a marginal and a conditional normal distribution. Use the result from part (e) in problem 8 to expand

$$(\underset{\sim}{x} - \underset{\sim}{\mu})'\Sigma^{-1}(\underset{\sim}{x} - \underset{\sim}{\mu}) = \begin{pmatrix} \underset{\sim}{x}_1 - \underset{\sim}{\mu}_1 \\ \underset{\sim}{x}_2 - \underset{\sim}{\mu}_2 \end{pmatrix}' \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}^{-1} \begin{pmatrix} \underset{\sim}{x}_1 - \underset{\sim}{\mu}_1 \\ \underset{\sim}{x}_2 - \underset{\sim}{\mu}_2 \end{pmatrix}$$

in an appropriate way to show that the multivariate normal density function is a product of the density functions for the marginal distribution of $\underset{\sim}{X}_2$ and a conditional distribution. What happens when $\Sigma_{12} = 0$, i.e. when $\underset{\sim}{X}_1$ and $\underset{\sim}{X}_2$ are uncorrelated?

For additional practice you could do problems 3.6, 3.8, 3.9, 3.10, 3.14, 3.16 at the end of Chapter 3 and problems 4.1, 4.3, 4.4, 4.5, 4.14, 4.15, 4.16 , 4.17 at the end of Chapter 4. Problem 4.8 gives a trivial example of a non-normal bivariate distribution with normal marginal distributions. We will consider a more substantial example on the next assignment. Do not hand in these additional problems; answers will be given on the solution sheet for this assignment.