Stat 501 **Homework 1 Solutions** **Spring 2005**

1. (a) $|\sum| = \lambda_1\lambda_2\lambda_3 = 144$

   (b) $trace(\sum) = \lambda_1 + \lambda_2 + \lambda_3 = 20$

   (c)

$$\begin{aligned} V(\underset{\sim}{e}_1' \underset{\sim}{x}) &= \underset{\sim}{e}_1' \sum \underset{\sim}{e}_1 = \underset{\sim}{e}_1'(\lambda_1 \underset{\sim}{e}_1) \\ &= \lambda_1 \underset{\sim}{e}_1' \underset{\sim}{e}_1 \\ &= \lambda_1 = 12 \end{aligned}$$

   by definition, the eigenvector $\underset{\sim}{e}_1$ has the properties $\sum \underset{\sim}{e}_1 = \lambda_1 \sum \underset{\sim}{e}_1$ and $\sum \underset{\sim}{e}_1' \sum \underset{\sim}{e}_1 = 1$.

   (d) $\sum = \lambda_1 \underset{\sim}{e}_1 \underset{\sim}{e}_1' + \lambda_2 \underset{\sim}{e}_2 \underset{\sim}{e}_2' + \lambda_3 \underset{\sim}{e}_3 \underset{\sim}{e}_3'$

$$= \begin{pmatrix} 8 & 2 & 2 \\ 2 & 6 & 4 \\ 2 & 4 & 6 \end{pmatrix}$$

   (e) $\sum^{-1} = \left(\frac{1}{\lambda_1}\right) \underset{\sim}{e}_1 \underset{\sim}{e}_1' + \left(\frac{1}{\lambda_2}\right) \underset{\sim}{e}_2 \underset{\sim}{e}_2' + \left(\frac{1}{\lambda_3}\right) \underset{\sim}{e}_3 \underset{\sim}{e}_3'$

$$= \begin{pmatrix} 5/36 & -1/36 & -1/36 \\ -1/36 & 11/36 & -7/36 \\ -1/36 & -7/36 & 11/36 \end{pmatrix}$$

   (f) $\sum^{-1/2} = \frac{1}{\sqrt{\lambda_1}} \underset{\sim}{e}_1 \underset{\sim}{e}_1' + \frac{1}{\sqrt{\lambda_2}} \underset{\sim}{e}_2 \underset{\sim}{e}_2' + \frac{1}{\sqrt{\lambda_3}} \underset{\sim}{e}_3 \underset{\sim}{e}_3'$

$$= \begin{pmatrix} 0.3684 & -0.0399 & -0.0399 \\ -0.0399 & 0.5178 & -0.1893 \\ -0.0399 & -0.1893 & 0.5178 \end{pmatrix}$$

   (g) Let $\underset{\sim}{\mu} = (\mu_1, \mu_2, \mu_3)'$ denote the mean vector for $\underset{\sim}{X}$.
   Then

$$E(\underset{\sim}{Y}) = \begin{bmatrix} \underset{\sim}{e}_1' \underset{\sim}{\mu} \\ \underset{\sim}{e}_2' \underset{\sim}{\mu} \\ \underset{\sim}{e}_3' \underset{\sim}{\mu} \end{bmatrix} = \begin{bmatrix} \frac{6}{\sqrt{3}} \\ \frac{3}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, V(\underset{\sim}{Y}) = \begin{bmatrix} 12 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

   (h) $\underset{\sim}{y}' \underset{\sim}{y} = (\Gamma' \underset{\sim}{X})'(\Gamma' \underset{\sim}{X}) = \underset{\sim}{X}'(\Gamma')'\Gamma' \underset{\sim}{X} = \underset{\sim}{X}' \Gamma \Gamma' \underset{\sim}{X} = \underset{\sim}{X}' \underset{\sim}{X}$ because Γ is an orthogonal matrix, that is, $\Gamma\Gamma' = \Gamma'\Gamma = I$.

(i) i. Yes, use result 2A.9.(e) on page 97 of the text and $|I| = 1$. Then, $|\sum_y| = |\Gamma' \sum_X \Gamma| = |\Gamma'||\sum_X||\Gamma| = |\Gamma'||\Gamma||\sum_X| = |\Gamma'\Gamma||\sum_X| = |I||\sum_X|$. This makes intuitive sense because $|\sum_y|$ and $|\sum_X|$ are proportional to the square of the volume of the same ellipse. Rotations do not change volume.

ii. Yes, use result 2A.12.(c) on page 98 of the text. Then $trace(\sum_y) = trace(\Gamma' \sum_X \Gamma) = trace(\Gamma\Gamma' \sum_X) = trace(I \sum_X) = trace(\sum_X)$.

iii. No, $\sum_Y$ is a diagonal matrix for any $\sum_X$ regardless of whether or not $\sum_X$ is a diagonal matrix.

iv. Yes, note that $\Gamma \sum_Y \Gamma'$ is the spectral decomposition of $\sum_X$. Then, $\sum_X = \Gamma \sum_Y \Gamma'$, $\sum_X^{-1} \Gamma \sum_Y^{-1} \Gamma'$ and $\Gamma' \sum_X \Gamma = \Gamma'(\Gamma \sum_Y^{-1} \Gamma')\Gamma = (\Gamma'\Gamma) \sum_Y^{-1} (\Gamma'\Gamma) = I \sum_Y^{-1} I = \sum_Y^{-1}$.
Consequently,

$$\begin{aligned}(\underset{\sim}{Y} - \underset{\sim}{\mu}_Y)' \textstyle\sum_Y^{-1} (\underset{\sim}{Y} - \underset{\sim}{\mu}_Y) &= (\Gamma' \underset{\sim}{X} - \Gamma' \underset{\sim}{\mu}_X)' \Gamma' \textstyle\sum_X^{-1} \Gamma (\Gamma' \underset{\sim}{X} - \Gamma' \underset{\sim}{\mu}_X) \\ &= [\Gamma' (\underset{\sim}{X} - \underset{\sim}{\mu}_X)]' \Gamma' \textstyle\sum_X^{-1} \Gamma [\Gamma' (\underset{\sim}{X} - \underset{\sim}{\mu}_X)] \\ &= (\underset{\sim}{X} - \underset{\sim}{\mu}_X)' (\Gamma')' \Gamma' \textstyle\sum_X^{-1} \Gamma \Gamma' (\underset{\sim}{X} - \underset{\sim}{\mu}_X) \\ &= (\underset{\sim}{X} - \underset{\sim}{\mu}_X)' (\Gamma\Gamma') \textstyle\sum_X^{-1} (\Gamma\Gamma') (\underset{\sim}{X} - \underset{\sim}{\mu}_X) \\ &= (\underset{\sim}{X} - \underset{\sim}{\mu}_X)' \textstyle\sum_X^{-1} (\underset{\sim}{X} - \underset{\sim}{\mu}_X)\end{aligned}$$

v. Yes,

$$\begin{aligned}(\underset{\sim}{Y} - \underset{\sim}{\mu}_Y)' (\underset{\sim}{y} - \underset{\sim}{\mu}_Y) &= (\Gamma' \underset{\sim}{X} - \Gamma' \underset{\sim}{\mu}_X)' (\Gamma' \underset{\sim}{X} - \Gamma' \underset{\sim}{\mu}_X) \\ &= [\Gamma' (\underset{\sim}{X} - \underset{\sim}{\mu}_X)]' [\Gamma' (\underset{\sim}{X} - \underset{\sim}{\mu}_X)] \\ &= (\underset{\sim}{X} - \underset{\sim}{\mu}_X)' (\Gamma')' \Gamma' (\underset{\sim}{X} - \underset{\sim}{\mu}_X) \\ &= (\underset{\sim}{X} - \underset{\sim}{\mu}_X)' (\underset{\sim}{X} - \underset{\sim}{\mu}_X)\end{aligned}$$

because $\Gamma\Gamma' = I$ Hence, both Euclidean distance and Mahalanobis distance are invariant to rotations.

2. (a) $f(x_1, x_2) = \frac{1}{2\pi |\sum|^{1/2}} exp \left[-\frac{1}{2} (\underset{\sim}{X} - \underset{\sim}{\mu}_X)' \textstyle\sum^{-1} (\underset{\sim}{X} - \underset{\sim}{\mu}_X) \right]$

$$|\textstyle\sum| = 36, \ \textstyle\sum^{-1} = \frac{1}{36} \begin{pmatrix} 9 & 3 \\ 3 & 5 \end{pmatrix}$$

$$f(x_1, x_2) = \frac{1}{2\pi\sqrt{36}} exp \left[-\frac{1}{2} \left(\underset{\sim}{X} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right)' \left(\frac{1}{36} \begin{pmatrix} 9 & 3 \\ 3 & 5 \end{pmatrix} \right) \left(\underset{\sim}{X} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) \right]$$

$$= \frac{1}{12\pi} exp\left[-\frac{5}{8}[(\frac{x_1-2}{\sqrt{5}})^2 + \frac{2}{\sqrt{5}}(\frac{x_1-2}{\sqrt{5}})(\frac{x_2-1}{3}) + (\frac{x_2-1}{3})^2]\right]$$

(b) $\lambda_1 = 7 + \sqrt{13} = 10.6056$, $\lambda_2 = 7 - \sqrt{13} = 3.3944$,

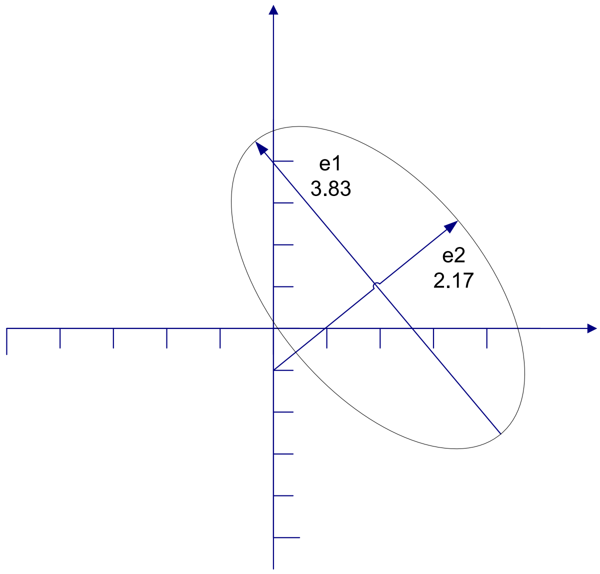
$$\underset{\sim}{e}_1 = \begin{bmatrix} 0.4719 \\ -0.8817 \end{bmatrix}, \underset{\sim}{e}_2 = \begin{bmatrix} 0.8817 \\ 0.4719 \end{bmatrix}$$

(c) $|\sum| = 36$, $trace(\sum) = 14$

(d) The ellipse given by

$$\frac{1}{36}\left[9(x_1-2)^2 + 6(x_1-2)(x_2-1) + 5(x_2-1)^2\right] = \chi^2_{(2).50} = 1.386$$

(e) $\sqrt{\chi^2_{(2)0.5}}\sqrt{\lambda_1} = 3.83$
$\sqrt{\chi^2_{(2)0.5}}\sqrt{\lambda_2} = 2.17$



(f) $area = \pi\chi^2_{(2).50}|\sum|^{1/2} = (\pi)(1.39)(6) = 27.95$

(g) $area = \pi\chi^2_{(2).05}|\sum|^{1/2} = (\pi)(5.99)(6) = 112.91$

3. $\underset{\sim}{\mu} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \sum = \begin{bmatrix} 5/36 & 1/12 \\ 1/12 & 1/4 \end{bmatrix}$

4. (a) (i),(v) display independent pairs of random variables, you simply have to check if the covariance is zero in each case.

(b) $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim N\left(\begin{bmatrix} -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}\right)$

(c) A bivariate normal distribution with mean vector

$$\begin{bmatrix} -3 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} [2]^{-1}(X_3 + 1) = \begin{bmatrix} (-\frac{X_3}{2} - 4) \\ 1 \end{bmatrix}$$

and covariance matrix

$$\begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} -1 \\ 0 \end{bmatrix} [2]^{-1} \begin{bmatrix} -1 & 0 \end{bmatrix} = \begin{bmatrix} 7/2 & 0 \\ 0 & 5 \end{bmatrix}$$

(d) $\rho_{12} = 0$ and $\rho_{12\cdot 3} = 0$

(e) Since X_2 are independent to $\begin{bmatrix} X_1 \\ X_3 \end{bmatrix}$,

$X_2 | x_1, x_3 \sim N(1, 5)$

5. (a) $Z_1 \sim N(-26, 77)$

(b) $\begin{bmatrix} Z_1 \\ Z2 \end{bmatrix} = \begin{bmatrix} 3 & -5 & 0 & 1 \\ 2 & 0 & -1 & 3 \end{bmatrix} \underset{\sim}{X} \sim N\left(\begin{bmatrix} -26 \\ -5 \end{bmatrix}, \begin{bmatrix} 77 & 59 \\ 59 & 89 \end{bmatrix}\right)$

(c) A bivariate normal distribution with mean vector

$$\begin{aligned} \begin{bmatrix} -4 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -2 \end{bmatrix} \frac{1}{9}(1 - 0) &= \begin{bmatrix} -4 \\ 1 \end{bmatrix} + \begin{bmatrix} 2/9 \\ -2/9 \end{bmatrix} \\ &= \begin{bmatrix} -34/9 \\ 7/9 \end{bmatrix} \end{aligned}$$

and covariance matrix

$$\begin{aligned} &\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 2 \\ -2 \end{bmatrix} \frac{1}{9} \begin{bmatrix} 2 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 4/9 & -4/9 \\ -4/9 & 4/9 \end{bmatrix} = \begin{bmatrix} 32/9 & 22/9 \\ 22/9 & 32/9 \end{bmatrix} \end{aligned}$$

(d) $\rho_{14\cdot 23} = \frac{5/2}{\sqrt{7/2 \times 7/2}} = \frac{5}{7} = 0.714$

6. (a) $cov(\underset{\sim}{X} - \underset{\sim}{Y}, \underset{\sim}{X} + \underset{\sim}{Y}) = cov(\underset{\sim}{X}, \underset{\sim}{X}) + cov(\underset{\sim}{X}, \underset{\sim}{Y}) - cov(\underset{\sim}{X}, \underset{\sim}{Y}) - cov(\underset{\sim}{Y}, \underset{\sim}{Y})$

$$= \sum_1 - \sum_2 = \begin{bmatrix} 4 & -4 \\ -4 & 0 \end{bmatrix}$$

(b) No, because $cov(\underset{\sim}{X} - \underset{\sim}{Y}, \underset{\sim}{X} + \underset{\sim}{Y})$ is not a matrix of zeros.

(c) Start with the fact that $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ are independent. Then from result 4.5(c) on page 160 of the text,

$$\begin{bmatrix} \underset{\sim}{X} \\ \underset{\sim}{Y} \end{bmatrix} \sim N\left(\begin{bmatrix} 2 \\ 6 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 8 & -2 & 0 & 0 \\ -2 & 4 & 0 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix} \right)$$

then from result 4.3 on page 157 of the text

$$\begin{bmatrix} \underset{\sim}{X} - \underset{\sim}{Y} \\ \underset{\sim}{X} + \underset{\sim}{Y} \end{bmatrix} = \begin{bmatrix} I & -I \\ I & I \end{bmatrix} \begin{bmatrix} \underset{\sim}{X} \\ \underset{\sim}{Y} \end{bmatrix}$$

$$\sim N\left(\begin{bmatrix} -1 \\ 2 \\ 5 \\ 10 \end{bmatrix}, \begin{bmatrix} 12 & 0 & 4 & -4 \\ 0 & 8 & -4 & 0 \\ 4 & -4 & 12 & 0 \\ -4 & 0 & 0 & 8 \end{bmatrix} \right)$$

7. (a) $d(\underset{\sim}{p}, \underset{\sim}{q})$ for any A. $d(\underset{\sim}{p}, \underset{\sim}{q}) = [(\underset{\sim}{p}, \underset{\sim}{q})^{'} A(\underset{\sim}{p}, \underset{\sim}{q})]^{1/2} > 0$ for any $(\underset{\sim}{p} - \underset{\sim}{q})$ if A is positive definite and it always true that $d(\underset{\sim}{p}, \underset{\sim}{p}) = 0$. Now we only have to consider property (iv). It easy to show that (iv) is true when A is both positive definite and symmetric. Then, the spectral decomposition matrix $A = \Gamma\Lambda\Gamma^{'}$, exists, where Λ is a diagonal matrix containing the eigenvalues for A and the columns of Γ are the corresponding eigenvector. Then, $A^{1/2} = \Gamma \begin{bmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_p} \end{bmatrix} \Gamma^{'}$ is also symmetric and positive definite and $A^{1/2}A^{1/2} = A$. Define $\underset{\sim}{d} = A^{1/2}(\underset{\sim}{r} - \underset{\sim}{q})$ and $\underset{\sim}{a}^{'} = (\underset{\sim}{p} - \underset{\sim}{r})^{'} A^{1/2}$.

Use the Cauchy-Schwarz inequality (page 80 in the text) to show that

$$\begin{aligned} [(\underset{\sim}{p} - \underset{\sim}{r})^{'} A(\underset{\sim}{r} - \underset{\sim}{q})]^2 &= (\underset{\sim}{a}^{'} \underset{\sim}{d})^2 \\ &\leq (\underset{\sim}{a}^{'} \underset{\sim}{a})(\underset{\sim}{d}^{'} \underset{\sim}{d}) \\ &= [d(\underset{\sim}{p}, \underset{\sim}{q})]^2 [d(\underset{\sim}{r}, \underset{\sim}{q})]^2. \end{aligned}$$

Then,

$$\begin{aligned} [d(\underset{\sim}{p}, \underset{\sim}{q})]^2 &= (\underset{\sim}{p} - \underset{\sim}{q})^{'} A(\underset{\sim}{p} - \underset{\sim}{q}) \\ &= (\underset{\sim}{p} - \underset{\sim}{r} + \underset{\sim}{r} - \underset{\sim}{q})^{'} A(\underset{\sim}{p} - \underset{\sim}{r} + \underset{\sim}{r} - \underset{\sim}{q}) \\ &= (\underset{\sim}{p} - \underset{\sim}{r})^{'} A(\underset{\sim}{p} - \underset{\sim}{r}) + (\underset{\sim}{r} - \underset{\sim}{q})^{'} A(\underset{\sim}{p} - \underset{\sim}{r}) \\ &\quad + (\underset{\sim}{p} - \underset{\sim}{r})^{'} A(\underset{\sim}{r} - \underset{\sim}{q}) + (\underset{\sim}{r} - \underset{\sim}{q})^{'} A(\underset{\sim}{r} - \underset{\sim}{q}) \\ &= [d(\underset{\sim}{p}, \underset{\sim}{r})]^2 + [d(\underset{\sim}{r}, \underset{\sim}{q})]^2 + 2(\underset{\sim}{p} - \underset{\sim}{q})^{'} A(\underset{\sim}{r} - \underset{\sim}{q}) \end{aligned}$$

$$
\begin{aligned}
&\leq [d(\underset{\sim}{p},\underset{\sim}{r})]^2 + [d(\underset{\sim}{r},\underset{\sim}{q})]^2 + 2d(\underset{\sim}{p},\underset{\sim}{r})d(\underset{\sim}{r},\underset{\sim}{q}) \\
&= [d(\underset{\sim}{p},\underset{\sim}{r}) + d(\underset{\sim}{r},\underset{\sim}{q})]^2
\end{aligned}
$$

Hence, (iv) is established for any positive definite symmetric matrix A.

(b) Show that (iv) holds for any positive definite matrix A,
note that $[d(\underset{\sim}{p},\underset{\sim}{q})]^2 = (\underset{\sim}{p}-\underset{\sim}{q})^{'}A(\underset{\sim}{p}-\underset{\sim}{q})$ is a scalar, so it is equal to its transpose.

Hence, $(\underset{\sim}{p}-\underset{\sim}{q})^{'}A(\underset{\sim}{p}-\underset{\sim}{q}) = [(\underset{\sim}{p}-\underset{\sim}{q})^{'}A(\underset{\sim}{p}-\underset{\sim}{q})]^{'} = (\underset{\sim}{p}-\underset{\sim}{q})^{'}A^{'}(\underset{\sim}{p}-\underset{\sim}{q})$

Then,

$$
\begin{aligned}
[d(\underset{\sim}{p},\underset{\sim}{q})]^2 &= \frac{1}{2}\left[(\underset{\sim}{p}-\underset{\sim}{q})^{'}A(\underset{\sim}{p}-\underset{\sim}{q}) + (\underset{\sim}{p}-\underset{\sim}{q})^{'}A(\underset{\sim}{p}-\underset{\sim}{q})\right] \\
&= (\underset{\sim}{p}-\underset{\sim}{q})^{'}\left[\frac{1}{2}(A+A^{'})\right](\underset{\sim}{p}-\underset{\sim}{q})
\end{aligned}
$$

satisfies (iv) by the previous argument because $\frac{1}{2}(A+A^{'})$ is symmetric and positive definite. Consequently, A dose not have to be symmetric, but it must be positive definite for $(\underset{\sim}{p}-\underset{\sim}{q})^{'}A(\underset{\sim}{p}-\underset{\sim}{q})$ to satisfy properties (i)-(iv) of a distance measure. If A was not positive definite, $\underset{\sim}{p}-\underset{\sim}{q} \neq \underset{\sim}{0}$ would exist for which $(\underset{\sim}{p}-\underset{\sim}{q})^{'}A(\underset{\sim}{p}-\underset{\sim}{q}) = 0$ and this would violate condition (ii).

(c) Simply apply the triangle inequality to each component of $(\underset{\sim}{p}-\underset{\sim}{q})$, *i.e.*,

$$
\begin{aligned}
d(\underset{\sim}{p},\underset{\sim}{q}) &= max(|p_1-q_1|,|p_2-q_2|) \\
&= max(|p_1-r_1+r_1-q_1|,|p_2-r_2+r_2-q_2|) \\
&\leq max(|p_1-r_1|+|r_1-q_1|,|p_2-r_2|+|r_2-q_2|) \\
&\leq max\big(|p_1-r_1|+max(|r_1-q_1|,|r_2-q_2|), \\
&\quad |p_2-r_2|+max(|r_1-q_1|,|r_2-q_2|)\big) \\
&\leq max(|p_1-r_1|,|p_2-r_2|)+max(|r_1-q_1|,|r_2-q_2|) \\
&= d(\underset{\sim}{p},\underset{\sim}{r})+d(\underset{\sim}{r},\underset{\sim}{q})
\end{aligned}
$$

8. (a) i.

$$
\begin{aligned}
&\begin{bmatrix} I_{q\times q} & B \\ 0_{r\times q} & I_{r\times r} \end{bmatrix}\begin{bmatrix} I_{q\times q} & -B \\ 0_{r\times q} & I_{r\times r} \end{bmatrix} \\
&= \begin{bmatrix} I_{q\times q} & -I_{q\times q}B_{q\times r}+B_{q\times r}I_{r\times r} \\ 0_{r\times q}I_{q\times q}+I_{r\times r}0_{r\times q} & I_{r\times r} \end{bmatrix} \\
&= \begin{bmatrix} I_{q\times q} & 0_{q\times r} \\ 0_{r\times q} & I_{r\times r} \end{bmatrix} = I_{(q+r)\times(q+r)}
\end{aligned}
$$

ii.

$$\begin{bmatrix} I_{r\times r} & 0_{r\times q} \\ B & I_{q\times q} \end{bmatrix} \begin{bmatrix} I_{r\times r} & 0_{r\times q} \\ -B & I_{q\times q} \end{bmatrix}$$
$$= \begin{bmatrix} I_{r\times r} - 0_{r\times q}B_{q\times r} & I_{r\times r}0_{r\times q} + 0_{r\times q}I_{q\times q} \\ B_{q\times r}I_{r\times r} - I_{q\times q}B_{q\times r} & B_{q\times r}0_{r\times q} + I_{q\times q}I_{q\times q} \end{bmatrix}$$
$$= \begin{bmatrix} I_{r\times r} & 0_{r\times q} \\ 0_{q\times r} & I_{q\times q} \end{bmatrix} = I_{(r+q)\times(r+q)}$$

(b) The determinant of a block triangle matrix can be obtained by simply multiplying the determinants of the diagonal blocks. Consequently,

$$\begin{vmatrix} I_{q\times q} & B \\ 0 & I_{r\times r} \end{vmatrix} = |I_{q\times q}||I_{r\times r}| = 1$$

and

$$\begin{vmatrix} I_{r\times r} & 0 \\ B & I_{q\times q} \end{vmatrix} = |I_{r\times r}||I_{q\times q}| = 1$$

To formly prove the latter case, start with the martix with $q = 1$ and expand across the first row to compute the determinant. The determinant of each minor in the expansion is multiplied by zero, except for the determinant of the $r \times r$ identity matrix which is multiplied by one. Cosequently, the determinant is one. Using an inductive proof, assume the result is true for any q and use a similar expansion acroos the first row to show that it is also true when the upper left block has dimesion q+1.

(c) $$\begin{bmatrix} I_{q\times q} & -A_{12}A_{22}^{-1} \\ 0_{r\times q} & I_{r\times r} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} I_{q\times q} & 0_{q\times r} \\ -A_{22}^{-1}A_{21} & I_{r\times r} \end{bmatrix}$$
$$= \begin{bmatrix} I_{q\times q}A_{11} - A_{12}A_{22}^{-1}A_{21} & I_{q\times q}A_{12} - A_{12}A_{22}^{-1}A_{22} \\ 0_{r\times q}A_{11} + I_{r\times r}A_{21} & 0_{r\times q}A_{12} + I_{r\times r}A_{22} \end{bmatrix} \begin{bmatrix} I_{q\times q} & 0_{q\times r} \\ -A_{22}^{-1}A_{21} & I_{r\times r} \end{bmatrix}$$
$$= \begin{bmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & 0 \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} I_{q\times q} & 0_{q\times r} \\ -A_{22}^{-1}A_{21} & I_{r\times r} \end{bmatrix}$$
$$= \begin{bmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & 0_{q\times r} \\ 0_{r\times q} & A_{22} \end{bmatrix}$$

(d) from 8(c),

$$\begin{vmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & 0_{q\times r} \\ 0_{r\times q} & A_{22} \end{vmatrix} = \begin{vmatrix} I_{q\times q} & -A_{12}A_{22}^{-1} \\ 0_{r\times q} & I_{r\times r} \end{vmatrix} \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} \begin{vmatrix} I_{q\times q} & 0_{q\times r} \\ -A_{22}^{-1}A_{21} & I_{r\times r} \end{vmatrix}$$

$LHS = |A_{22}||A_{11} - A_{12}A_{22}^{-1}A_{21}|$ for $|A_{22}| \neq 0$

$RHS = 1 \times |A| \times 1 = |A|$ from 8(b).

Hence, $|A| = |A_{11} - A_{12}A_{22}^{-1}A_{21}|$ for $|A_{22}| \neq 0$

(e) Take inverse on both sides of 8(c), then

$$\begin{bmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & 0_{q\times r} \\ 0_{r\times q} & A_{22} \end{bmatrix}^{-1} = \left(\begin{bmatrix} I_{q\times q} & -A_{12}A_{22}^{-1} \\ 0_{r\times q} & I_{r\times r} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} I_{q\times q} & 0_{q\times r} \\ -A_{22}^{-1}A_{21} & I_{r\times r} \end{bmatrix} \right)^{-1}$$

$$= \begin{bmatrix} I_{q\times q} & -A_{12}A_{22}^{-1} \\ 0_{r\times q} & I_{r\times r} \end{bmatrix}^{-1} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1} \begin{bmatrix} I_{q\times q} & 0_{q\times r} \\ -A_{22}^{-1}A_{21} & I_{r\times r} \end{bmatrix}^{-1}$$

then we easily get

$$A^{-1} = \begin{bmatrix} I_{q\times q} & -A_{12}A_{22}^{-1} \\ 0_{r\times q} & I_{r\times r} \end{bmatrix} \begin{bmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & 0_{q\times r} \\ 0_{r\times q} & A_{22} \end{bmatrix}^{-1} \begin{bmatrix} I_{q\times q} & 0_{q\times r} \\ -A_{22}^{-1}A_{21} & I_{r\times r} \end{bmatrix}$$

from result of 8(a), we know that

$$\begin{bmatrix} I_{q\times q} & -A_{12}A_{22}^{-1} \\ 0_{r\times q} & I_{r\times r} \end{bmatrix} = \begin{bmatrix} I_{q\times q} & 0_{q\times r} \\ -A_{22}^{-1}A_{21} & I_{r\times r} \end{bmatrix}^{-1}$$

$$\begin{bmatrix} I_{q\times q} & 0_{q\times r} \\ -A_{22}^{-1}A_{21} & I_{r\times r} \end{bmatrix} = \begin{bmatrix} I_{q\times q} & -A_{12}A_{22}^{-1} \\ 0_{r\times q} & I_{r\times r} \end{bmatrix}^{-1}$$

9. Define $\quad \underset{\sim}{X} = \begin{pmatrix} \underset{\sim}{X}_1 \\ \underset{\sim}{X}_2 \end{pmatrix} \quad \underset{\sim}{\mu} = \begin{pmatrix} \underset{\sim}{\mu}_1 \\ \underset{\sim}{\mu}_2 \end{pmatrix} \quad \sum = \begin{pmatrix} \sum_{11} & \sum_{12} \\ \sum_{21} & \sum_{22} \end{pmatrix}$

$$f_{\underset{\sim}{X}}(\underset{\sim}{x}) = \frac{1}{(2\pi)^{p/2}|\sum|^{1/2}} exp\left[-\frac{1}{2}(\underset{\sim}{X} - \underset{\sim}{\mu})' \textstyle\sum^{-1}(\underset{\sim}{X} - \underset{\sim}{\mu})\right]$$

$$= \frac{1}{(2\pi)^{q/2}(2\pi)^{r/2}|\sum_{22}|^{1/2}|\sum_{11} - \sum_{12}\sum_{22}^{-1}\sum_{21}|^{1/2}} exp\left[-\frac{1}{2}(\underset{\sim}{X} - \underset{\sim}{\mu})' \textstyle\sum^{-1}(\underset{\sim}{X} - \underset{\sim}{\mu})\right]$$

Expand the quadratic form

$$(\underset{\sim}{X} - \underset{\sim}{\mu})' \textstyle\sum^{-1}(\underset{\sim}{X} - \underset{\sim}{\mu}) = \begin{pmatrix} (\underset{\sim}{X}_1 - \underset{\sim}{\mu}_1) \\ (\underset{\sim}{X}_2 - \underset{\sim}{\mu}_2) \end{pmatrix}' \begin{pmatrix} \sum_{11} & \sum_{12} \\ \sum_{21} & \sum_{22} \end{pmatrix}^{-1} \begin{pmatrix} (\underset{\sim}{X}_1 - \underset{\sim}{\mu}_1) \\ (\underset{\sim}{X}_2 - \underset{\sim}{\mu}_1) \end{pmatrix}$$

$$= \begin{pmatrix} (\underset{\sim}{X}_1 - \underset{\sim}{\mu}_1)' & (\underset{\sim}{X}_2 - \underset{\sim}{\mu}_2)' \end{pmatrix} \begin{bmatrix} I_{q\times q} & 0_{q\times r} \\ -\sum_{22}^{-1}\sum_{21} & I_{r\times r} \end{bmatrix} \begin{bmatrix} (\sum_{11} - \sum_{12}\sum_{22}^{-1}\sum_{21}) & 0_{q\times r} \\ 0_{r\times q} & \sum_{22}^{-1} \end{bmatrix}$$

$$\times \begin{bmatrix} I_{q\times q} & -\sum_{12}\sum_{22}^{-1} \\ 0_{r\times q} & I_{r\times r} \end{bmatrix} \begin{pmatrix} (\underset{\sim}{X}_1 - \underset{\sim}{\mu}_1) \\ (\underset{\sim}{X}_2 - \underset{\sim}{\mu}_1) \end{pmatrix}$$

$$= \begin{bmatrix} (\underset{\sim}{X}_1' - \underset{\sim}{\mu}_1') - (\underset{\sim}{X}_2' - \underset{\sim}{\mu}_2')\sum_{22}^{-1}\sum_{21}) & (\underset{\sim}{X}_2' - \underset{\sim}{\mu}_2') \end{bmatrix} \begin{bmatrix} (\sum_{11} - \sum_{12}\sum_{22}^{-1}\sum_{21}) & 0_{q\times r} \\ 0_{r\times q} & \sum_{22}^{-1} \end{bmatrix}$$

$$\times \begin{bmatrix} (\underset{\sim}{X}_1 - \underset{\sim}{\mu}_1) - \sum_{12} \sum_{22}^{-1} (\underset{\sim}{X}_2 - \underset{\sim}{\mu}_2) \\ (\underset{\sim}{X}_2 - \underset{\sim}{\mu}_2) \end{bmatrix}$$

$$= [(\underset{\sim}{X}_1 - \underset{\sim}{\mu}_1) - \sum_{12} \sum_{22}^{-1} (\underset{\sim}{X}_2 - \underset{\sim}{\mu}_2)]' (\sum_{11} - \sum_{12} \sum_{22}^{-1} \sum_{21})^{-1}$$

$$\times [(\underset{\sim}{X}_1 - \underset{\sim}{\mu}_1) - \sum_{12} \sum_{22}^{-1} (\underset{\sim}{X}_2 - \underset{\sim}{\mu}_2)] + (\underset{\sim}{X}_2 - \underset{\sim}{\mu}_2)' \sum_{22}^{-1} (\underset{\sim}{X}_2 - \underset{\sim}{\mu}_2)$$

$$= B_1 + B_2$$

$$f_{\underset{\sim}{X}}(\underset{\sim}{x}) = \frac{1}{(2\pi)^{q/2}(2\pi)^{r/2} |\sum_{22}|^{1/2} |\sum_{11} - \sum_{12} \sum_{22}^{-1} \sum_{21}|^{1/2}} exp \left[-\frac{1}{2} (\underset{\sim}{X} - \underset{\sim}{\mu})' \sum^{-1} (\underset{\sim}{X} - \underset{\sim}{\mu}) \right]$$

$$= \left(\frac{1}{(2\pi)^{q/2} |\sum_{11} - \sum_{12} \sum_{22}^{-1} \sum_{21}|^{1/2}} \right) \left(\frac{1}{(2\pi)^{r/2} |\sum_{22}|^{1/2}} \right) exp \left[-\frac{1}{2} (B_1 + B_2) \right]$$

$$= \frac{1}{(2\pi)^{q/2} |\sum_{11} - \sum_{12} \sum_{22}^{-1} \sum_{21}|^{1/2}} exp \left[-\frac{1}{2} B_1 \right] \left(\frac{1}{(2\pi)^{r/2} |\sum_{22}|^{1/2}} \right) exp \left[-\frac{1}{2} B_2 \right]$$

$$= f_{\underset{\sim}{X}_1 | x_2} (\underset{\sim}{x}_1) f_{\underset{\sim}{X}_2} (\underset{\sim}{x}_2)$$

If $\underset{\sim}{X}_1$ and $\underset{\sim}{X}_2$ are uncorrelated, then $f_{\underset{\sim}{X}_1 | X_2 = x_2} (\underset{\sim}{x}_1) = f_{\underset{\sim}{X}_1} (\underset{\sim}{x}_1)$

Therefore, $f_{\underset{\sim}{X} | x}(\underset{\sim}{x}) = f_{\underset{\sim}{X}_1} (\underset{\sim}{x}_1) f_{\underset{\sim}{X}_2} (\underset{\sim}{x}_2)$ imply $\underset{\sim}{X}_1$ and $\underset{\sim}{X}_2$ are independent.

The review of properties for partitioned matrices in problem 8 was done to give you the tools to factor the density for a multivariate normal distribution as a product of a marginal density and a conditional density. You may find some of these martix properties to be useful in future work.