STA 6557 Exam 1

April 5, 2022

This is a show your work, open book, open notes 70 minute exam, partially covering Chapters 2,3 and 4 from your textbook. Good luck!

EXERCISE 1. *Y_n is a sequence of random vectors in $\mathbb{R}^m$. Prove that if $Y_n \xrightarrow{P} Y$, then $Y_n \to_d Y$.*

EXERCISE 2. *Give an example of a random variables X_n $(n \geq 1), X$ on $(\mathbb{R}, \mathcal{B})$ such that $X_n \to_d X$, yet $P_{X_n}(B)$ does not converge to $P_X(B)$ for all Borel sets $B \in \mathcal{B}$.*

EXERCISE 3. *Assume X_j, $j \geq 1$, are i.i.d. real-valued, with $EX_j = \mu$, $\text{var}(X_1) = \sigma^2 > 0$, $EX_1^4 < \infty$. What is the asymptotic distribution of $\sqrt{n}(\ln s_n - \ln \sigma)$, where s_n^2 is the sample variance of $X_j, j = 1, \ldots, n$?*

EXERCISE 4. *Show that the real projective space $\mathbb{R}P^m$ is an analytic manifold of real dimension m, that can be embedded in the space of space $S(m+1, \mathbb{R})$ of real symmetric matrices endowed with the Euclidean distance d_0 given by $d_0^2(A, B) = Tr((A-B)(A-B)^t) = Tr((A-B)^2)$.*

EXERCISE 5. *a. Show that the unit sphere $\mathbb{S}^2 = \{x \in \mathbb{R}^3, \|x\|^2 = 1\}$ is an 2-dimensional manifold, and its tangent space $T\mathbb{S}^2$ is a 4 dimensional manifold.*

b. Consider the discrete random object X on $\mathbb{S}^2$ with $Pr(X = e_a) = p_a, p_1 + p_2 + p_3 = 1, p_a > 0, \forall a = 1, 2, 3$, where (e_1, e_2, e_3) is the standard basis of $\mathbb{R}^3$. Show that the X has an extrinsic mean and find the extrinsic mean of X in terms of p_1 and p_2.