Oriented projective shape analysis

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Abstract. The pinhole camera is the ubiquitous model for well-focused imaging systems. This model describes how points in three dimensions are projected onto the camera's two-dimension image plane which represents a digital image, for instance. The geometric features of this projection have classically been described in terms of projective geometry. This framework is physically unrealistic in the sense that one ignores directional information, i.e. it is not assumed that the scene being imagined lies in front of the camera. It has been noted in the computer vision literature that this is a problem and in fact results in greater sensitivity to measurement error. We take this directional information into account and develop the notion of oriented projective shape, and oriented projective shape space. Simulation studies show that the resulting extrinsic statistical techniques for image data have greater statistical power than comparable statistical techniques which ignore directional information. Here, 3D oriented projective shape analysis is made possible using 3D reconstructions from digital camera images.

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1 Introduction

Digital image data plays a vital role in engineering, the mathematical sciences, the biomedical sciences, and many other scientific and technological fields. Much of this image data comes from a security or cell-phone camera. For these types of image acquisition systems, knowledge about camera calibration and scene structure is often unknown. In this uncalibrated camera setting one cannot correct an image for perspective effects associated with the placement of the imaging system relative to the object of the interest. These perspective effects are however relatively easy to describe once one adopts the pinhole camera model. This camera model is known to

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be a good geometric approximation to many well-focused imaging systems (see Ma, et al. (2006), Hartley and Zisserman (2004)[2]).

Under the ideal pinhole camera the perspective projection of a point p, with Euclidean coordinates (X, Y, Z) (regarded as affine coordinate of the 3D projective point $[X : Y : Z : 1]$) onto the image plane, has Euclidean coordinates (x, y) (affine coordinates of the 2D projective point $[x : y : 1]$) as defined by the following relation in affine coordinates:

$$\lambda \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}.$$

where $\lambda = Z > 0$ is the unknown for an uncalibrated camera, and f is the focal distance.

At this point in the modeling process, it is very common to treat the homogenous coordinates as being unique up to a non-zero scalars when for practical purposes, they are really unique up to positive scalars. Under the classical assumption homogenous coordinates become points in projective spaces, albeit of different dimensions, and perspective projection model may now be cast in terms of an elegant projective geometric framework. We briefly review relevant concepts from projective geometry.

In projective geometry two non-zero vectors x and y in $(m + 1)$-dimensional numerical space $\mathbb{R}^{m+1}$ are equivalent if they differ by a non-zero scalar multiple. The equivalence class of $x \in \mathbb{R}^{m+1} \setminus \{0\}$ is labeled $[x]$. The set of all such equivalence classes is the projective space $P(\mathbb{R}^{m+1})$ associated with $\mathbb{R}^{m+1}$

$$P(\mathbb{R}^{m+1}) = \{[x];\ x \in \mathbb{R}^{m+1} \setminus \{0\}\}.$$

and is often denoted as $\mathbb{R}P^m$. The projective space $\mathbb{R}P^m$ topologically an m-dimensional unit sphere $\mathbb{S}^m$ with the antipodal points identified, and can be represented as a disjoint union of ordinary and ideal projective points;

$$\mathbb{R}P^m = \underbrace{\left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_m \\ 1 \end{bmatrix} \in \mathbb{R}P^m \right\}}_{\text{ordinary points}} \bigcup \underbrace{\left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_m \\ 0 \end{bmatrix} \in \mathbb{R}P^m \right\}}_{\text{ideal points}}$$

In this framework for instance, a vanishing point in a two-dimensional image corresponds to an (ideal) point in $\mathbb{R}P^2$. The downside to introducing projective geometry is that points behind the image plane are all virtual images of points in front of the camera. If we instead treat the homogenous coordinates as being unique up to a positive scalar, then we in effect assume that the scene being imagined lies in front of the camera and the perspective projection model is now described using oriented projective geometry; references include Stolfi(1991)[12].

In oriented projective geometry two non-zero vectors x and y in $\mathbb{R}^{m+1}$ are equivalent if they differ by a positive scalar multiple. The equivalence class of $x \in \mathbb{R}^{m+1} \setminus \{0\}$ is labeled $\overrightarrow{[x]}$. The set of all such equivalence classes is the oriented projective space

$\overrightarrow{P}(\mathbb{R}^{m+1})$ associated with $\mathbb{R}^{m+1}$

$$\overrightarrow{P}(\mathbb{R}^{m+1}) = \{\overrightarrow{[x]};\ x \in \mathbb{R}^{m+1} \setminus \{0\}\}$$

and is also identified with $\mathbb{S}^m$, set of vectors of Euclidean norm one in $\mathbb{R}^{m+1}$, via $\overrightarrow{[x]} \to \frac{1}{\|x\|}x$. One option for the choice of a distance between two points $\overrightarrow{[x]}$ and $\overrightarrow{[y]}$ in $\overrightarrow{P}(\mathbb{R}^{m+1})$ is the chord distance $d_0(\frac{1}{\|x\|}x, \frac{1}{\|y\|}y)$, where d_0 is the Euclidean distance in $\mathbb{R}^{m+1}$.

2 Oriented projective shape space

We first review concepts of projective transformations and projective shape space, as a starting point for our discussion of oriented projective transformations and oriented projective shape space. Below is a summary pertinent results (see Patrangenaru and Mardia (2005)[5], Munk et al(2008)[6], Paige et al(2005)[7], Qiu et al(2019)[11], Patrangenaru and Ellingson(2015)[9]).

A k-ad is an ordered list of k labeled points in $\mathbb{R}^m$; it can be also regarded as a k-ad in $\mathbb{R}P^m$, via the standard affine embedding of $\mathbb{R}^m$ in $\mathbb{R}P^m$ given by

$$x = (x_1, \dots, x_m) \to [x_1 : \dots : x_m : 1] = [(x_1 \dots x_m 1)^T].$$

Note that interchangeably, as deemed necessary, in this paper a point in $\mathbb{R}^d$ is regarded as column or row vector. Two k-ads of points in $\mathbb{R}^m$ have the same projective shape if they differ by a projective transformation of $\mathbb{R}^m$. Recall that

Definition 2.1. A projective transformation $g = g_P$ of $\mathbb{R}P^m$ is the projective map associated with a nonsingular matrix $P \in GL(m+1, \mathbb{R})$ and its action on $\mathbb{R}P^m$

$$g_P([x]) = g([x_1 : \dots : x_{m+1}]) = [P(x_1 \cdots x_{m+1})^T]$$

The projective transformations of $\mathbb{R}P^m$ form a group, denoted by $PGL(m)$.

Definition 2.2. An ordered list of $m+2$ labeled points in $\mathbb{R}P^m$ is said to form a projective frame (basis) if they span $\mathbb{R}P^m$. An ordered list of $k \geq m+2$ labeled projective points in $\mathbb{R}P^m$ are said to be in general position if the first $m+2$ of these points form a projective frame. $G\,(k, m)$ is the space of all k-ads in general position in $\mathbb{R}P^m$.

Definition 2.3. The projective shape of a k-ad $X = ([x_1], \dots, [x_k]) \in G\,(k, m)$ is the orbit of X under the action α of $PGL(m)$ on $G\,(k, m)$, given by

$$\alpha(g_P, X) = (g_P([x_1]), \dots, g_P([x_k])).$$

Definition 2.4. The projective shape space, $P\Sigma_m^k$, is the space of projective shapes of all k-ads in $\mathbb{R}P^m$ in general position

$$P\Sigma_m^k = G\,(k, m)\,/PGL(m).$$

Theorem 2.1. *$P\Sigma_m^k$ is a manifold, which is homeomorphic with*

$$\underbrace{\mathbb{R}P^m \times \mathbb{R}P^m \times \cdots \times \mathbb{R}P^m}_{q \text{ copies}} = (\mathbb{R}P^m)^q$$

where

$$q = k - m - 2.$$

We now extend the above concepts to the oriented projective setting, as first described in Stolfi(1991)[12], and further discussed in Lazebnik(2002)[3], and define novel concepts of oriented projective shape and oriented projective shape space.

Definition 2.5. An oriented projective transformation $g = g_P$ of $\overrightarrow{P}(\mathbb{R}^{m+1}) \equiv \mathbb{S}^m$ is a transformation associated with matrix $P \in GL^+(m+1, \mathbb{R})$, and is given by

$$g(\overrightarrow{[(x_1 \cdots x_{m+1})^T]}) = \overrightarrow{[P(x_1 \cdots x_{m+1})^T]}$$

where $GL^+(m+1, \mathbb{R})$ is the group of $(m+1) \times (m+1)$ matrices with positive determinant. $OPGL(m)$ is the group of oriented projective transformations.

Definition 2.6. An ordered list of $m+2$ labeled points in $\overrightarrow{P}(\mathbb{R}^{m+1}) \equiv \mathbb{S}^m$ is said to form an oriented projective frame (basis) if they span $\overrightarrow{P}(\mathbb{R}^{m+1})$, and the last point in the configuration is in the positive hull of the first $m+1$ points. An oriented k-ad is a k-ad such that the first $m+2$ of its points form an oriented projective frame. The set of all oriented k-ads in $\overrightarrow{P}(\mathbb{R}^{m+1})$ is denoted by $G^+(k, m)$.

Definition 2.7. The oriented projective shape $\overrightarrow{[X]}$ of an oriented k-ad $X = (\overrightarrow{[x_1]}, \ldots, \overrightarrow{[x_k]}) \in G^+(k, m)$ is the orbit of that k- ad under the action of $OPGL(m)$ given by

$$g_P(\overrightarrow{[x_1]}, \ldots, \overrightarrow{[x_k]}) = (\overrightarrow{[Px_1]}, \ldots, \overrightarrow{[Px_k]}), P \in GL^+(m+1, \mathbb{R}).$$

The oriented projective shape space, $OP\Sigma_m^k$, is the space of oriented projective shapes of oriented k-ads

$$OP\Sigma_m^k = G^+(k, m) / OPGL(m)$$

where $G^+(k, m)$ is the space of all oriented k-ads.

3 Oriented Projective Coordinates

We first review projective coordinates as a starting point for introducing oriented projective coordinates. Recalling from Mardia and Patrangenaru(2005)[5],

Definition 3.1. The standard projective basis is

$$([e_1], \ldots, [e_{m+1}], [e_1 + \cdots + e_{m+1}])$$

where

$$(e_1, \ldots, e_{m+1})$$

is the standard basis $\mathbb{R}^{m+1}$

Theorem 3.1. *Given two projective bases*

$$(p_1, \dots, p_{m+2})$$

and

$$(q_1, \dots, q_{m+2})$$

there is a unique projective transformation g defined for a $P \in GL(m+1, \mathbb{R})$ as

$$g([x_1 : \dots : x_{m+1}]) = [P(x_1, \cdots, x_{m+1})^T]$$

such that

$$g(p_j) = [Pp_j] = q_j$$

for $j = 1, \dots, m+2$

Corollary 3.2. *Given a k-ad in general position $(p_1, \dots, p_k), p_j = [x_j], j = 1, \dots, q$. Since $(p_1, \dots, p_{m+2})$ is a projective basis, there exists a unique projective transformation g defined for a $P \in GL(m+1, \mathbb{R})$ so that*

$$g(p_j) = [Px_j] = [e_j]$$

for $j = 1, \dots, m+1$ and

$$g(p_{m+2}) = [Px_{m+2}] = [e_1 + \dots + e_{m+1}]$$

Now we are ready to define the projective coordinates of a k-ad as first described in Patrangenaru (1999)[8].

Definition 3.2. Suppose that k-ad $X = (p_1, \dots, p_k), p_j = [x_j], j = 1, \dots, q$ is in general position, and let the projective transformation $g = g_P \in PGL(m), P \in GL(m+1, \mathbb{R})$ be such that

$$g(p_j) = [Px_j] = [e_j]$$

for $j = 1, \dots, m+1$ and

$$g(p_{m+2}) = [Px_{m+2}] = [e_1 + \dots + e_{m+1}]$$

will necessarily map the remaining $k - m - 2$ landmarks into a point in $(\mathbb{R}P^m)^{k-m-2}$ and these points are the projective coordinates of the k-ad X.

Now we consider analogous definitions and results for the oriented projective shape space case.

Definition 3.3. The standard oriented projective basis is

$$(\overrightarrow{[e_1]}, \dots, \overrightarrow{[e_{m+1}]}, \overrightarrow{[e_1 + \dots + e_{m+1}]})$$

where

$$(e_1, \dots, e_{m+1})$$

is the standard basis $\mathbb{R}^{m+1}$

Note that this basis has a positive orientation relative to $\mathbb{R}^{m+1}$ since the determinant of the matrix $(e_1, \dots, e_{m+1})$ is positive. In the approach of Stolfi(1991)[12]

$$(e_1, \dots, e_{m+1})$$

is thought of as a simplex and is termed *the canonical m dimensional simplex*, the oriented projective point

$$\overrightarrow{[e_1 + \cdots + e_{m+1}]}$$

is known as the unit point and the signature of

$$(\overrightarrow{[e_1]}, \dots, \overrightarrow{[e_{m+1}]}, \overrightarrow{[e_1 + \cdots + e_{m+1}]})$$

is denoted as

$$+ + \cdots +$$

since the unit point is generated from

$$\overrightarrow{[e_1]}, \dots, \overrightarrow{[e_{m+1}]}$$

as a linear combination with all positive coefficients so that the unit point is in the interior of the simplex

$$(\overrightarrow{[e_1]}, \dots, \overrightarrow{[e_{m+1}]}).$$

Following definition 7, we have

Proposition 3.3. *An oriented projective basis of $\overrightarrow{P}\mathbb{R}^{m+1}$ is an ordered set of labeled points $(p_1, \dots, p_{m+2}), p_j = \overrightarrow{[x_j]}, j = 1, \dots, m+2$, with $(x_1, \dots, x_{m+1})$ being positively oriented, and $x_{m+2} = \alpha_1 x_1 + \cdots + \alpha_1 x_{m+1}, \alpha_j > 0, \forall j = 1, \dots, m+1$.*

Theorem 3.4. *(Stolfi(1991)[12], Chapter 12) Given two oriented projective bases*

$$(p_1, \dots, p_{m+2})$$

and

$$(q_1, \dots, q_{m+2})$$

there is a unique oriented projective transformation $g = g_P \in OPGL(m), P \in GL^+(m+1, \mathbb{R})$ such that

$$g\,(p_j) = \overrightarrow{[Pp_j]} = q_j$$

for $j = 1, \dots, m+2$

Corollary 3.5. *Given an oriented k-ad $(p_1, ..., p_k), p_j = \overrightarrow{[x_j]}, j = 1. \dots, k$, there exists a unique oriented projective transformation $g = g_P \in OPGL(m)$ defined for a $P \in GL^+(m+1, \mathbb{R})$ so that*

$$g\,(p_j) = \overrightarrow{[Px_j]} = \overrightarrow{[e_j]}$$

for $j = 1, \dots, m+1$ and

$$g\,(p_{m+2}) = \overrightarrow{[Pp_{m+2}]} = \overrightarrow{[e_1 + \cdots + e_{m+1}]}$$

The oriented projective coordinates of an oriented k-ad $X = (p_1, ..., p_k), p_j = [\overrightarrow{x_j}], j = 1. \dots, k$ are now defined in the following manner: let g be the oriented projective transformation, given in Corollary 3.5, then $(g(p_{m+3}), \dots, g(p_k) \in (\mathbb{S}^m)^{k-m-2}$ are the *oriented projective coordinates* of the oriented k-ad $(p_1, ..., p_k)$.

Corollary 3.6. *$OP\Sigma_m^k$ is a manifold which is homeomorphic with $(\mathbb{S}^m)^q$ where, as before,*

$$q = k - m - 2.$$

Consider, for example, the k-ad

$$X = \begin{bmatrix} 69 & 591 & 626 & 69 & 344 \\ 53 & 33 & 402 & 430 & 322 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

and let

$$U = [x_1, x_2, x_3] = \begin{bmatrix} 69 & 591 & 626 \\ 53 & 33 & 402 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\begin{aligned} x_4 = U \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} &= [x_1, x_2, x_3] \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 69 & 591 & 626 \\ 53 & 33 & 402 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1286 \\ 488 \\ 3 \end{bmatrix} \end{aligned}$$

So that now our augmented k-ad is

$$\tilde{X} = \begin{bmatrix} 69 & 591 & 626 & 1286 & 69 & 344 \\ 53 & 33 & 402 & 488 & 430 & 322 \\ 1 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}$$

and

$$\begin{aligned} U^{-1}\tilde{X} &= \begin{bmatrix} 69 & 591 & 626 \\ 53 & 33 & 402 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 69 & 591 & 626 & 1286 & 69 & 344 \\ 53 & 33 & 402 & 488 & 430 & 322 \\ 1 & 1 & 1 & 3 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 1 & 1.068\,3 & 0.523\,79 \\ 0 & 1 & 0 & 1 & -1.086\,2 & -0.278\,60 \\ 0 & 0 & 1 & 1 & 1.018\,0 & 0.754\,81 \end{bmatrix} \end{aligned}$$

so that our oriented projective coordinate is

$$\left(\begin{bmatrix} 1.0683 \\ -1.0862 \\ 1.0180 \end{bmatrix}, \begin{bmatrix} 0.52379 \\ -0.27860 \\ 0.75481 \end{bmatrix} \right)$$

Normalizing these vectors to have unit length yields the spherical representation of oriented projective coordinate as

$$\left(\begin{bmatrix} 0.58301 \\ -0.59282 \\ 0.55557 \end{bmatrix}, \begin{bmatrix} 0.545\,58 \\ -0.290\,19 \\ 0.786\,21 \end{bmatrix} \right)$$

4 Power study

The Face Research Lab in the Institute of Neuroscience and Psychology at the University of Glasgow has published composite portraits of men and women from around to world which approximate the "average face" of each gender by country. The average white American male and female faces are shown below using the 9 facial landmarks, as measured in freeware program ImageJ, which we consider in our power study. For details on how to obtain an average face digital image using means of landmark configurations see Patrangenaru and Patrangenaru(2004)[10].

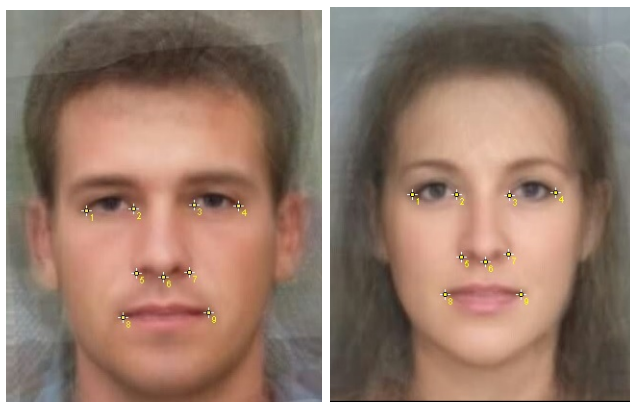

Figure 1: Average of white American male face images (left) and of white American female face images(right), based on oriented shape means of landmark configurations

In our power study we simulated from two populations generated by random perspective transformations of the randomly perturbed landmarks. For each simulation run these perturbations were bivariate normal realizations with zero mean and a covariance matrix taken to be the 2 by 2 sample covariance of the landmark points. For each set of gender landmarks total of 4 bivariate normal realizations were used. This is because we assume certain symmetries in these perturbations in that landmarks 1 and 4 had the same perturbation, landmarks 2 and 3 had the same perturbation, landmarks 5, 6 and 7 have their own perturbation, and landmarks 8 and 9 have their own landmark.

Chapter 11 of Corke (2011)[1] describes the geometric model of perspective image creation for the pinhole camera model in detail and provides MATLAB routines for generating perspective images which we used in our simulations. In reference to figure 11.5 of Corke(2011)[1] and the ensuing discussion, numerical experimentation indicated that using a focal value of 0.008 meters and a 1024×1024 pixel array of 10^{-6} meter square pixels worked well. The original landmarks coordinates for each scene were translated so that their centroid or geometric center coincided with the principal point or centroid of the pixel array. For each scene we randomly and uniformly varied the mathematical camera pose angles in horizontal (x) and vertical (y) directions, rotations about the optical axis (z), and randomly translated the camera model in $\mathbb{R}^3$ based upon the values of three independent standard normal realizations. This resulted in more than 90% of the randomly generated landmark points all of whom were located in front of the camera, with a value of $z > 0$. If any of the nine simulated landmark points ended up behind the camera then all nine landmark points were discarded and another draw was obtained.

In our Monte Carlo study we had 1,000 runs in which independent samples of size 100 for each set of gender landmarks was performed. Extrinsic tests of differences where used for each of the 1,000 Monte Carlo runs and each of the two methods considered. Given the large sizes used in each run, asymptotic cutoff values were used. The Monte Carlo estimates of power were as follows:

Projective	0.279
Oriented Projective	0.887

Here oriented projective statistic was computed using Bhattacharya's two-sample test method for extrinsic means; see section 6.3 of Patrangenaru and Ellingson (2015).

5 3D current and future work

Statistical analyses in oriented projective shape space are likely to have more statistical power than analogous procedures in developed in classical projective shape space. One possible application would involve tests for differences in three-dimensional surfaces like those shown below.

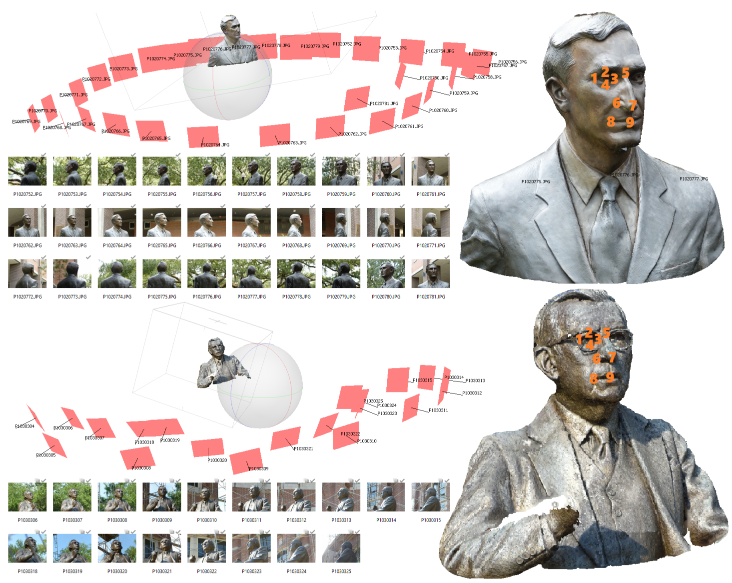

Figure 2: Retrieving and comparing 3D face oriented projective shapes from digital camera images

This is part could have particularly important application, say, in the navigation systems for self-driving cars particularly when they are operating under less than ideal illumination conditions.

For future work, we plan to build textured surfaces of living individuals, using the same RGB correlation method used for statues surface reconstruction.

5.1 Simultaneous confidence regions for the projective shape change between two FSU presidents statues

The figures 3 are simultaneous confidence regions for the mean 3D projective shape face configurations change for the two FSU presidents, presented above. The data consists 30 photos taken from one president and 18 photos taken from the other president, as described in figure 2. In this case, a matched pairs test cannot be used since the number of samples from one president differs from that of the other president. We utilize the landmarks 1,4,5,6,8 to construct the projective frame and compute sample means based on the landmarks 2,3,7,9 in figure 2 to get the confidence region for the projective shape change. For these four landmarks, the the point (0,0,0) is not in the 12 confidence intervals of the associate affine coordinates of the projective shape change. Therefore there is a significant mean projective shape change.

It would be interesting to define 3D oriented projective shape change and to plot simultaneous confidence regions for the mean 3D oriented projective shape change in future. Comparing the computational times for similar tasks in the projective shape analysis vs oriented projective shape analysis are also deferred to future work.

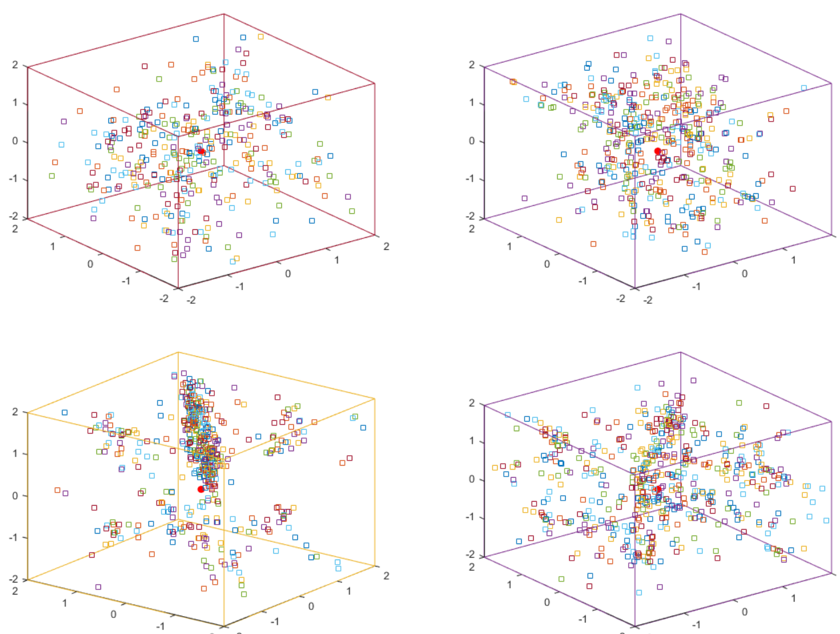

Figure 3: Simultaneous confidence regions for the marginal axial components of mean projective shape changes in affine coordinates for the configuration selected for the two presidents.

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